TMA4115 Matematikk 3

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Spring 2010

Lecture 8: Ode to ODE

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5th February 2010

Recap

- Solve inhomogeneous by:
  1. Solve homogeneous
  2. Find particular solution
  3. Solve IVP
- Constant coefficients + periodic forcing displays . . . unusual behaviour

Key Points

- Use solutions of homogeneous to "simplify" search for solution of inhomogeneous

Lessons From Pendula

Mathematical Models are

1. Idealised
2. Simplified
3. Limited validity

Strange Behaviour

1. Error in **Implementation**
   Fix it, and hope no-one notices.
2. Error in **Model**
   Find boundaries in model.
3. Error in **Reality**
   Publish and get a Nobel prize!

Form(ula) versus Function

Definition

A function $f\colon I \to \mathbb{R}$ is a
rule
assigning a real number to each number in I.

Example

1. $t \mapsto t^2$
2. $t \mapsto \sin(t)$
3. $\theta \mapsto x$–coordinate of point with angle θ
4. $t \mapsto t$–value of solution of

$$y'' + p(x)y' + q(x)y = r(x)$$

Unknown Knowns

Key Point

- $f(t)$ must be **uniquely** determined
- $f(t)$ need **not** be known

Known versus Determined
Value of solution of

$$y'' + (y')^2 + 1 = 0$$

at $x = \pi/2$?

$$0$$

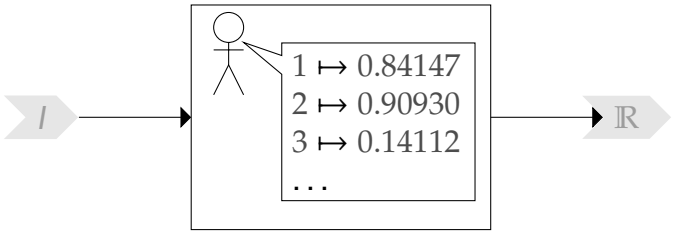
Impersonal Definition

- Function does not depend on **knowledge of user**
- Function does not depend on **exact implementation**

Example

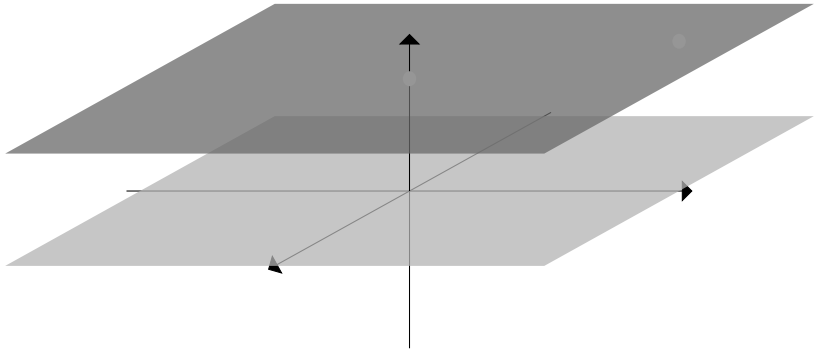
1. "$t \mapsto t$–value of solution of ODE" is fine
2. $2\sin(x)\cos(x)$ and $\sin(2x)$ are **same** function

Form(ula) versus Function



III : Implementation Is Irrelevant

Finding a Plane: the ODE Way



1. Find plane through origin.
2. Find point on plane.
3. Simplify search: look in "obvious" direction.

Spot of Geometry

- Plane through origin: $\{\lambda\vec{v} + \mu\vec{w}\}$
- Translated plane: $\{\vec{u} + \lambda\vec{v} + \mu\vec{w}\}$
- Lots of choices for $\vec{u}$.
- "Best" choice has $\vec{u}$ **orthogonal** to plane:

$$\vec{u}^{\mathsf{T}}\vec{v} = 0, \qquad \vec{u}^{\mathsf{T}}\vec{w} = 0$$

- Makes problem simpler by **artifically** imposing additional constraints.
- $\Big($ General nonsense $\Longrightarrow$ no loss of generality $\Big)$

Simplifying Problems

$$y'' + y = \tan\left(\log\left(\frac{\sin(x)}{2+\cos(x)}\right)\right)$$

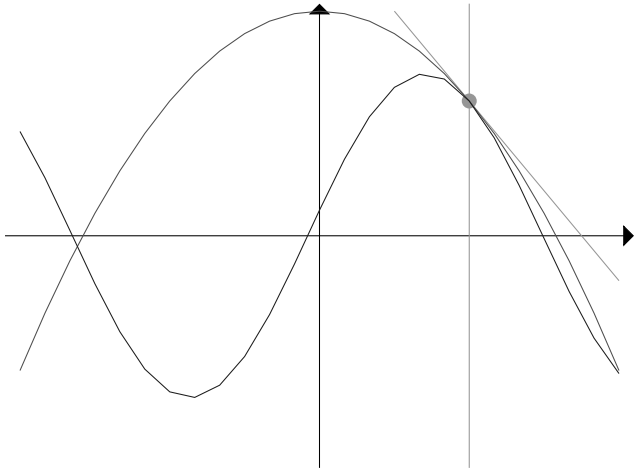
Solutions to Homogeneous: $\sin(x)$, $\cos(x)$.

Solution of Inhomogeneous? $y(x)$?

- $y(x)$ **fixed** by $y(x_0)$, $y'(x_0)$, **any** $x_0 \in I$.
- Have solution of **homogeneous** ODE through the **same** points:

$$c_{x_0}\sin(x) + d_{x_0}\cos(x)$$

In Pictures



Known Unknowns

Key Principle

Approximate unknown by **most useful** knowns.

- In this case, solutions of homogeneous ODE.
- Computation: polynomials
- Signal analysis: trigonomials
- Plotting: piecewise-linear
- ...

What's the Point?

$$y(x) = c_{x_0} \sin(x) + d_{x_0} \cos(x)$$

- only 100% accurate at x_0 could be bad elsewhere!
- but for **each** x_0 there's a unique (c_{x_0}, d_{x_0}) that works
- $x_0 \mapsto c_{x_0}$ and $x_0 \mapsto d_{x_0}$ are **functions**!

Conclusion

Have functions $c(x)$, $d(x)$ such that for **all** $x \in I$

$$y(x) = c(x)y_1(x) + d(x)y_2(x)$$
$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x)$$

So What?

Know $y(x)$

$\Longleftrightarrow$

Know $c(x)$ and $d(x)$

Bad Trade?

1 unknown function $\longleftrightarrow$ 2 unknown functions

Maybe we gain elsewhere...

The Trade

Conditions:

$$y(x) = c(x)y_1(x) + d(x)y_2(x) \quad (1)$$
$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x) \quad (2)$$

- But (**??**) is **not** the derivative of (**??**)!

Differentiating:

$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x) + c'(x)y_1(x) + d'(x)y_2(x)$$

Thus:

$$0 = c'(x)y_1(x) + d'(x)y_2(x)$$

Use The ODE, Luke

$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x)$$

Differentiating Again:

$$y''(x) = c(x)y_1''(x) + d(x)y_2''(x) + c'(x)y_1'(x) + d'(x)y_2'(x)$$

Substituting In:

$$\begin{aligned} y'' + p(x)y' + q(x)y &= r(x) \\ c(x)y_1''(x) + d(x)y_2''(x) + c'(x)y_1'(x) + d'(x)y_2'(x) & \\ +p(x)\Big(c(x)y_1'(x) + d(x)y_2'(x)\Big) & \\ +q(x)\Big(c(x)y_1(x) + d(x)y_2(x)\Big) &= r(x) \end{aligned}$$

Rearranging

$$\begin{aligned} y'' + p(x)y' + q(x)y &= r(x) \\ cy_1'' + dy_2'' + c'y_1' + d'y_2' & \\ +p\Big(cy_1' + dy_2'\Big) & \\ +q\Big(cy_1 + dy_2\Big) &= r \\ c\Big(y_1'' + py_1' + qy_1\Big) & \\ + d\Big(y_2'' + py_2' + qy_2\Big) & \\ c'y_1' + d'y_2' &= r \\ c'y_1 + d'y_2 &= 0 \end{aligned}$$

Solving

$$\begin{aligned} c'y_1'y_1 + d'y_2'y_1 &= ry_1 \\ c'y_1y_1' + d'y_2y_1' &= 0 \\ d'\Big(y_1y_2' - y_1'y_2\Big) &= ry_1 \end{aligned}$$

$$d = \int \frac{ry_1}{W(y_1, y_2)}\mathrm{d}x$$

$$c = -\int \frac{ry_2}{W(y_1, y_2)}\mathrm{d}x$$

Variation of Parameters

Particular Solution

$$y(x) = c(x)y_1(x) + d(x)y_2(x)$$
$$c(x) = -\int^x \frac{r(t)y_2(t)}{W(y_1, y_2)(t)}\mathrm{d}t$$
$$d(x) = \int^x \frac{r(t)y_1(t)}{W(y_1, y_2)(t)}\mathrm{d}t$$

Example

$$y'' - 2y' + y = \mathrm{e}^x \sin(x)$$

Homogeneous: e^x, $x\mathrm{e}^x$
Wronskian: $W(y_1, y_2) = \mathrm{e}^x(\mathrm{e}^x + x\mathrm{e}^x) - \mathrm{e}^x x\mathrm{e}^x = \mathrm{e}^{2x}$
Variation of Parameters:

$$\begin{aligned} c(x) &= -\int^x \frac{\mathrm{e}^t \sin(t) t\mathrm{e}^t}{\mathrm{e}^{2t}}\mathrm{d}t \\ &= -\int^x t\sin(t)\mathrm{d}t \\ &= [t\cos(t)]^x - \int^x \cos(t)\mathrm{d}t \\ &= x\cos(x) - \sin(x) \end{aligned}$$

Example ctd.

$$\begin{aligned} d(x) &= \int^x \frac{\mathrm{e}^t \sin(t)\mathrm{e}^t}{\mathrm{e}^{2t}}\mathrm{d}t \\ &= \int^x \sin(t)\mathrm{d}t \\ &= -\cos(x) \end{aligned}$$

General Solution:

$$\Big(x\cos(x) - \sin(x)\Big)\mathrm{e}^x - \cos(x)x\mathrm{e}^x + a\mathrm{e}^x + bx\mathrm{e}^x$$
$$= -\sin(x)\mathrm{e}^x + a\mathrm{e}^x + bx\mathrm{e}^x$$

Always check!

Example

$$x^2y'' - 2x^{-1}y' + 2y = x^3\cos(x)$$

Homogeneous: x, x^2
Wronskian: $W(y_1, y_2) = x2x - x^2 = x^2$
Variation of Parameters:

$$\begin{aligned} c(x) &= -\int^x \frac{t\cos(t)t^2}{t^2}\mathrm{d}t \\ &= -\int^x t\cos(t)\mathrm{d}t \\ &= -x\sin(x) - \cos(x) \end{aligned}$$

Example ctd

$$\begin{aligned} d(x) &= \int^x \frac{t\cos(t)t}{t^2}\mathrm{d}t \\ &= \int^x \cos(t)\mathrm{d}t \\ &= \sin(x) \end{aligned}$$

General Solution:

$$-\Big(x\sin(x)+\cos(x)\Big)x + x^2\sin(x) + ax + bx^2 = -x\cos(x) + ax + bx^2$$

Summary

- ODEs have solutions
- No **guaranteed** method for finding
- Techniques for special cases
- Use with care!
- Use partial information to get more