TMA4115 Matematikk 3

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Lecture 7: Plane Sailing

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Recap

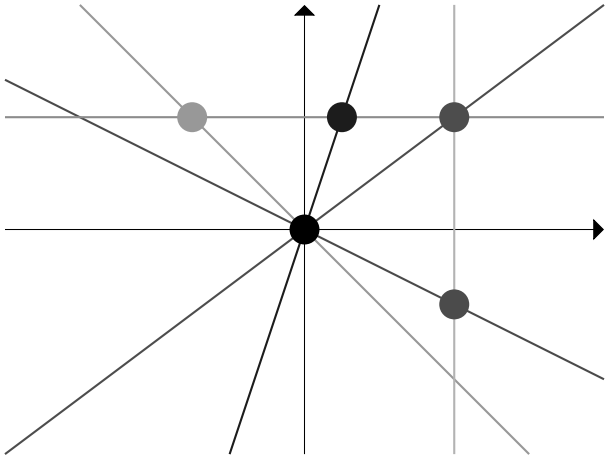
- Homogeneous ODEs:
  2 solutions $\Longrightarrow$ all solutions
- Inhomogeneous ODEs:
  Guess answer based on output.

Key Points

- Solve inhomogenous by solving homogeneous first
- Inhomogeneous with constant coefficients

How Many Points?

Question
How many points specify a line?



How Many Solutions?

Solutions of

$$y'' + p(x)y' + q(x)y = r(x)$$

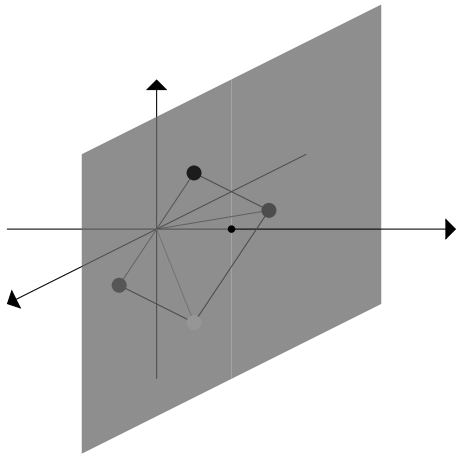
look like a **plane** in some **huge** space.

- Homogeneous $\iff r(x) = 0 \iff$ origin $\in$ plane
- Inhomogeneous $\iff r(x) \neq 0 \iff$ origin $\notin$ plane

Need 3 points to specify a plane.

Homogeneous $\implies$ Take one to be "origin" (i.e. 0)

Specifying a Plane



$$\text{point} = \text{point} + \big(\text{point} - \text{point}\big) + \big(\text{point} - \text{point}\big)$$

Specifying Points on a Plane

Plane: $\mathrm{P} \ni \vec{u}, \vec{v}, \vec{w}$

$$\begin{aligned}\mathrm{P} &= \left\{a\vec{u} + b\vec{v} + c\vec{w} : a + b + c = 1\right\} \\ &= \left\{\vec{u} + b(\vec{v} - \vec{u}) + c(\vec{w} - \vec{u}) : b, c \in \mathbb{R}\right\}\end{aligned}$$

Solutions: y_1, y_2, y_3 of

$$y'' + p(x)y' + q(x)y = r(x)$$

$$\begin{aligned}\text{All} &= \left\{ay_1 + by_2 + cy_3 : a + b + c = 1\right\} \\ &= \left\{y_1 + b(y_2 - y_1) + c(y_3 - y_1) : b, c \in \mathbb{R}\right\}\end{aligned}$$

Specifying Solutions of ODE

$$y'' + p(x)y' + q(x)y = r(x)$$

$$\begin{aligned}\text{All} &= \left\{ ay_1 + by_2 + cy_3 : a + b + c = 1 \right\} \\ &= \left\{ y_1 + b(y_2 - y_1) + c(y_3 - y_1) : b, c \in \mathbb{R} \right\}\end{aligned}$$

Question
Which way is best?

Answer
The **second** way!

The Second Way

Because: $y_2 - y_1$, $y_3 - y_2$ solutions of **homogeneous** ODE:

$$\begin{aligned}&(y_2 - y_1)'' + p(x)(y_2 - y_1)' + q(x)(y_2 - y_1) \\ &\qquad = y_2'' + p(x)y_2' + q(x)y_2 - y_1'' - p(x)y_1' - q(x)y_1 \\ &\qquad = r(x) - r(x) \\ &\qquad = 0\end{aligned}$$

So don't depend on $r(x)$.

Often want to allow $r(x)$ to vary:

- 3 solutions $\Longrightarrow$ 3 new each time
- $1 + 2$ solutions $\Longrightarrow$ 1 new each time

General Solution

ODE: $y'' + p(x)y' + q(x)y = r(x)$

Solutions: $y = y_p + y_h$

- y_p **particular** solution of **inhomogeneous** ODE
- y_h **general** solution of **homogeneous** ODE
- $\left(\text{so } y_h = c_1 y_1 + c_2 y_2 \right)$

Three Step Method for IVP:

1. Solve homogeneous ODE completely
2. Find **one** solution of inhomogeneous
3. Fit to initial conditions.

Examples

$$y'' + 2y' + y = \mathrm{e}^x$$

$\frac{1}{4}\mathrm{e}^x$ is a **particular** solution.

e^{-x} and $x\mathrm{e}^{-x}$ lin ind solutions of **homogeneous** ODE.

General Solution

$$\frac{1}{4}\mathrm{e}^x + a\mathrm{e}^{-x} + bx\mathrm{e}^{-x}$$

IVP: $y(0) = 5/4$, $y'(0) = 9/4$

$$\frac{1}{4}\mathrm{e}^x + \mathrm{e}^{-x} + x\mathrm{e}^{-x}$$

Examples

$$y'' - y = e^x$$

$\frac{1}{2}xe^x$ is a **particular** solution.

e^x and e^{-x} lin ind solutions of **homogeneous** ODE.

General Solution

$$\frac{1}{2}xe^x + ae^x + be^{-x}$$

IVP: $y(0) = 2$, $y'(0) = 3^1/_2$

$$\frac{1}{2}xe^x + 2e^x - e^{-x}$$

Examples

$$y'' - 2y' + y = e^x$$

$\frac{1}{2}x^2e^x$ is a **particular** solution.

e^x and xe^x lin ind solutions of **homogeneous** ODE.

General Solution

$$\frac{1}{2}x^2e^x + ae^x + bxe^x$$

IVP: $y(0) = 2$, $y'(0) = 3^1/_2$

$$\frac{1}{2}x^2e^x - e^x - xe^x$$

Examples

$$y'' + 2y' + y = x$$

$x - 2$ is a **particular** solution.

e^{-x} and xe^{-x} lin ind solutions of **homogeneous** ODE.

General Solution

$$x - 2 + ae^{-x} + bxe^{-x}$$

IVP: $y(2) = 0$, $y'(2) = 0$

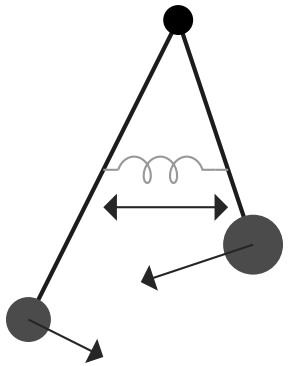
$$x - 2 + 2e^{2-x} - xe^{2-x}$$

Case Study: Coupled Pendula

Warning

position, y, function of **time**, t
Kreyzig considers **springs**

Case Study: Coupled Pendula



Coupled Pendula: The Equations

$$M_a L_a y_a'' = F_a = -M_a g \sin(y_a) - c L_a y_a' + k(y_b - y_a)$$
$$M_b L_b y_b'' = F_b = -M_b g \sin(y_b) - c L_b y_b' + k(y_a - y_b)$$

$$y_a'' + \frac{c}{M_a} y_a' + \left(\frac{g}{L_a} + \frac{k}{M_a L_a}\right) y_a = \frac{k}{M_a L_a} y_b$$
$$y_b'' + \frac{c}{M_b} y_b' + \left(\frac{g}{L_b} + \frac{k}{M_b L_b}\right) y_b = \frac{k}{M_b L_b} y_a$$

Simplifying Assumption: M_b very big.

$$y_b = \sin(\omega t)$$

Forced Oscillation

$$y'' + ay' + by = F \sin(\omega t)$$

Solutions of Homogeneous Equation:

1. $e^{-\lambda t}\cos(\nu t)$, $e^{-\lambda t}\sin(\nu t)$. Underdamped
2. $e^{-\lambda_1 t}$, $e^{-\lambda_2 t}$. Overdamped
3. $e^{-\lambda t}$, $te^{-\lambda t}$. Critical Damping

Trig in, Trig out: Try $c\sin(\omega t) + d\cos(\omega t)$:

$$\begin{aligned} &-\omega^2(c\sin(\omega t) + d\cos(\omega t)) \\ &\quad + a\omega(c\cos(\omega t) - d\sin(\omega t)) \\ &\quad + b(c\sin(\omega t) + d\cos(\omega t)) = F\sin(\omega t) \end{aligned}$$

$$(b - \omega^2)d + a\omega c = 0, \qquad (b - \omega^2)c - a\omega d = F$$

Forced Oscillation

$$(b - \omega^2)d + a\omega c = 0, \qquad (b - \omega^2)c - a\omega d = F$$

Has a solution **unless** $a\omega = 0$ **and** $\omega^2 = b$

- $a = 0$: No Friction.
- $\omega^2 = b$: Forcing period same as natural period.

Both highly unlikely!

But already ignored small stuff so don't need **equality** just **indistinguishable**.

Consider Coincidences: No Friction

$$y'' + \nu^2 y = F\sin(\omega t)$$

Solutions of Homogeneous Equation:

- $\cos(\nu t)$, $\sin(\nu t)$. No damping

Particular Solution:

$$\frac{F}{\nu^2 - \omega^2}\sin(\omega t)$$

General Solution:

$$\frac{F}{\nu^2 - \omega^2}\sin(\omega t) + c\sin(\nu t) + d\cos(\nu t)$$

Coincidences: No Friction, Special Start

$$y'' + \nu^2 y = F\sin(\omega t)$$

General Solution:

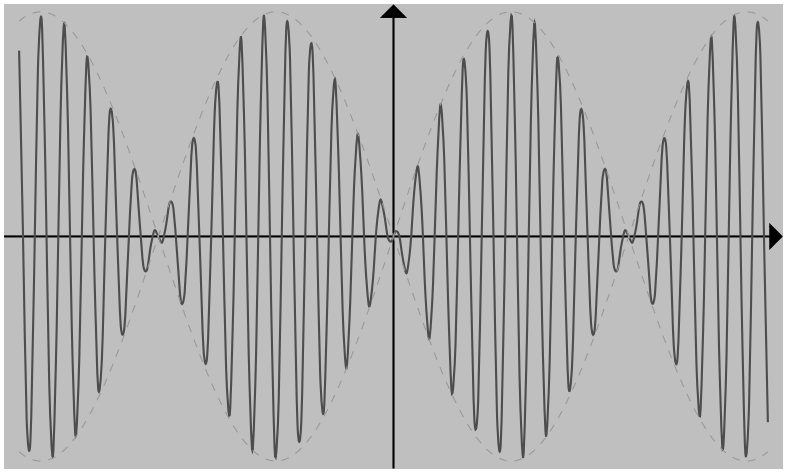
$$\frac{F}{\nu^2 - \omega^2}\sin(\omega t) + c\sin(\nu t) + d\cos(\nu t)$$

IVP: chosen to get

$$\frac{F}{\nu^2 - \omega^2}(\sin(\omega t) - \sin(\nu t))$$
$$= \frac{F}{2(\nu^2 - \omega^2)}\sin\left(\frac{\omega - \nu}{2}t\right)\cos\left(\frac{\omega + \nu}{2}t\right)$$

Slow variation and fast oscillation.

Beats



But best to **hear** it!

Coincidences: No Friction, Same Period

$$y'' + \omega^2 y = F\sin(\omega t)$$

Solutions of Homogeneous Equation:

- $\cos(\omega t)$, $\sin(\omega t)$.

Particular Solution:

$$\frac{F}{2\omega}t\cos(\omega t)$$

General Solution:

$$\frac{F}{2\omega}t\cos(\omega t) + c\sin(\omega t) + d\cos(\omega t)$$

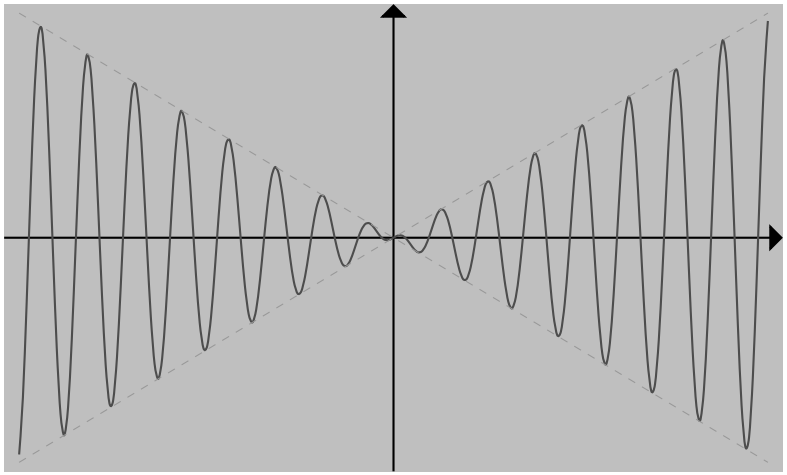
Resonance

Conclusions

1. Mathematical conclusion: Solution grows with time.
2. Physical conclusion: Model becomes inaccurate.
3. Engineering conclusion: Build a better bridge.

Observation: Dominant term is out of phase with forcing.

Resonance



But best to **see** it!

Summary

- Solve inhomogeneous IVP:
  1. Solve homogeneous
  2. Find particular solution
  3. Fit to conditions
- Constant coefficients with forcing:
  1. Friction $\Longrightarrow$ decay to follow force
  2. No friction $\Longrightarrow$ beats or resonance