TMA4115 Matematikk 3

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Lecture 6: The Grand Tour Continues

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Recap

Uniqueness

$$y'' + p(x)y' + q(x)y = 0$$
$$y(x_0) = K_0,\ y'(x_0) = K_1$$

Interval I:

1. $p(x)$, $q(x)$ continuous on I
2. $x_0 \in I$

Solution exists **and is unique**

Key Points

- Wronskian:

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

- Use Wronskian to test for linear independence
- Inhomogenous ODEs:

$$y'' + p(x)y' + q(x)y = r(x)$$

- Solve by intelligent guesswork

Testing Linear Independence

Tests
Solutions y_1 and y_2

1. Is $y_1 = 0$?
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
3. Is $y_1'(x_0) \neq 0$ and $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1$?

Yes to any $\Longrightarrow$ dependent **No to all** $\Longrightarrow$ independent

Examples

$$y'' - 2y' + y = 0$$

Solutions: $y_2(x) = e^x$, $y_1(x) = xe^x$
Question: Are these linearly independent?

1. Is $y_1 = 0$?
   No. $2e^2 \neq 0$.
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
   E.g. $x_0 = 1$: "Yes ($1e^1 \neq 0$) and No ($e^x \neq xe^x$)".
   E.g. $x_0 = 0$: "No and so move on."
3. Is $y_1'(x_0) \neq 0$ and $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1$?
   ($x_0 = 0$): "Yes ($0e^0 + e^0 \neq 0$) and No ($e^x \neq xe^x$)".

Conclusion: Linearly independent

Examples

$$y'' - 3x^{-1}y' + 3x^{-2}y = 0$$

Solutions: $y_1(x) = x$, $y_2(x) = x^3$
Question: Are these linearly independent?

1. Is $y_1 = 0$?
   No. $1 \neq 0$.
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
   E.g. $x_0 = 1$: "Yes ($1 \neq 0$) and No ($x^3 \neq x$)".
3. Is $y_1'(x_0) \neq 0$ and $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1$?
   E.g. ($x_0 = 1$): "Yes ($1 \neq 0$) and No ($x \neq 3x$)".

Conclusion: Linearly independent

Examples

$$y'' + 4y = 0$$

Solutions: $y_1(x) = \sin(2x)$, $y_2(x) = \sin(x)\cos(x)$
Question: Are these linearly independent?

1. Is $y_1 = 0$?
   No. $1 = \sin(2\pi/4) \neq 0$.
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
   E.g. $x_0 = \pi/4$: "Yes ($1 \neq 0$) and Yes ($\sin(x)\cos(x) = 1/2 \sin(2x)$)".
3. Is $y_1'(x_0) \neq 0$ and $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1$?
   E.g. ($x_0 = 0$): "Yes ($2 \neq 0$) and Yes ($\sin(x)\cos(x) = 2/4 \sin(2x)$)".

Conclusion: Not linearly independent
(aka linearly **dependent**)

Testing Linear Independence

1. Is $y_1 = 0$?
   Are $y_1(x_0) = 0$ and $y_1'(x_0) = 0$?
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
   $y_3 := y_2 - \frac{y_2(x_0)}{y_1(x_0)} y_1$ solution with $y_3(x_0) = 0$
   $y_3 = 0 \iff y_3'(x_0) = 0$
   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y_2'(x_0) = \frac{y_2(x_0)}{y_1(x_0)} y_1'(x_0)$
   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y_2'(x_0) y_1(x_0) - y_2(x_0) y_1'(x_0) = 0$
3. Is $y_1'(x_0) \neq 0$ and $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1$?
   $y_3 := y_2 - \frac{y_2'(x_0)}{y_1'(x_0)} y_1$ solution with $y_3'(x_0) = 0$
   $y_3 = 0 \iff y_3(x_0) = 0$
   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y_2(x_0) = \frac{y_2'(x_0)}{y_1'(x_0)} y_1(x_0)$
   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y_2(x_0) y_1'(x_0) - y_2'(x_0) y_1(x_0) = 0$

Testing Linear Independence

Claim

y_1 and y_2 not linearly independent
$\iff$
$y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0) = 0$
for **some** x_0

How can it be zero?

1. $y_1(x_0) = 0$ and $y_1'(x_0) = 0 \implies$ test 1 holds
2. $y_1(x_0) \neq 0 \implies$ test 2 holds
3. $y_1(x_0) \neq 0 \implies$ test 3 holds

Introducing . . . The Wronskian

Definition

*The **Wronskian** of two functions*
y_1, y_2
is

$$W(y_1, y_2) := y_1 y_2' - y_1' y_2$$

Examples

1. $W(x, x^2) = x 2x - x^2 = x^2$
2. $W(e^x, e^{-x}) = e^x(-e^{-x}) - e^x e^{-x} = -2$
3. $W(\cos(x), \sin(x)) = \cos(x)\cos(x) - (-\sin(x)\sin(x)) = 1$

Wronskian and Linear Independence

$$y'' + p(x)y' + q(x)y = 0$$

y_1 and y_2 solutions of ODE.
y_1, y_2 linearly independent
$\iff$
$W(y_1, y_2)(x) \neq 0$ for all x
$\iff$
$W(y_1, y_2)(x_0) \neq 0$ for some x_0

The Norwegian Connection

Really Weird Fact

$W(y_1, y_2)$ depends on ODE and **almost** not on y_1 and y_2!

$$y'' - 2y' + 2y = 0$$

Solutions: $ax + bx^2$

$$\begin{aligned} W(a_1x + b_1x^2, a_2x + b_2x^2) &= (a_1x + b_1x^2)(a_2 + 2b_2x) \\ &\quad - (a_1 + 2b_1x)(a_2x + b_2x^2) \\ &= a_1a_2x + (2a_1b_2 + a_2b_1)x^2 + 2b_1b_2x^3 \\ &\quad - a_1a_2x - (a_1b_2 + 2a_2b_1)x^2 - 2b_1b_2x^3 \\ &= (a_1b_2 - a_2b_1)x^2 \end{aligned}$$

Always a multiple of x^2!

Abel's Identity

$$\begin{aligned} W(y_1, y_2)' &= y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' \\ &= y_1(-py_2' - qy_2) - (-py_1' - qy_1)y_2 \\ &= -p(y_1y_2' - y_1'y_2) \\ &= -pW(y_1, y_2) \end{aligned}$$

1st Order Linear ODE!
Solution:

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)e^{-\int_{x_0}^{x} p(t)\,\mathrm{d}t}$$

Caution

- Wronskian makes sense for **any** functions
- Linear independence makes sense for **any** functions
- "Non-zero Wronskian $\iff$ Linear independent" only guaranteed for **solutions of an ODE**

$$\begin{aligned} W(x, \sin(x))(x) &= x\cos(x) - \sin(x) \\ W(x, \sin(x))(0) &= 0 \end{aligned}$$

But x, $\sin(x)$ **are** linearly independent.

The Next Step: Inhomogeneous ODEs

Introducing ...

$$y'' + p(x)y' + q(x)y = r(x)$$

2nd Order **Inhomogeneous** Linear ODE

Example

$$\begin{aligned} y'' + 2y' + y &= e^x \\ y'' - x^{-1}y' + x^{-2}y'' &= e^x \\ y'' - \sin(x)y' + (x)y &= x^2 - (x+1) \end{aligned}$$

Uniqueness Revisited

Uniqueness

$$y'' + p(x)y' + q(x)y = r(x)$$
$$y(x_0) = K_0,\ y'(x_0) = K_1$$

Interval I:

1. $p(x)$, $q(x)$, $r(x)$ continuous on I
2. $x_0 \in I$

Solution exists **and is unique**

Miracle Proof of Uniqueness

y_1, y_2 both solutions
$$\Longrightarrow$$
$y_1 - y_2$ solution of **homogeneous** ODE:

$$y'' + p(x)y' + q(x)y = 0$$
$$y(x_0) = 0,\ y'(x_0) = 0$$

$$\Longrightarrow$$
$$y_1 = y_2$$

Suggests: Homogeneous may provide **useful** information about inhomogeneous.

Solving ODEs: Strategy One: Just Guess

(Intelligently, of course)

Basic Principle

Put in (almost) what you want out.

Based on:

$$\begin{aligned}
\text{Polynomials} &\xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} \text{Polynomials}\\
\text{Exponentials} &\longrightarrow \text{Exponentials}\\
\text{Trig} &\longrightarrow \text{Trig}\\
f + g &\longrightarrow f' + g'\\
xf &\longrightarrow xf' + f
\end{aligned}$$

Examples

$$y'' + 2y' + y = e^x$$

Try e^x:

$$\text{LHS}: \ e^x + 2e^x + e^x = 4e^x$$

Try: $\frac{1}{4}e^x$.

$$\text{LHS}: \ \frac{1}{4}e^x + \frac{2}{4}e^x + \frac{1}{4}e^x = e^x$$

Hence: $\frac{1}{4}e^x$ is a solution.

(What about other solutions? Come back to that later.)

Examples

$$y'' - y = e^x$$

Try e^x:

$$\text{LHS}: \ e^x - e^x = 0$$

- What next?
- Use trick from homogeneous case: multiply by x!

Try xe^x:

$$\text{LHS}: \ xe^x + 2e^x - xe^x = 2e^x$$

Hence: $\frac{1}{2}xe^x$ is a solution.

Examples

$$y'' - 2y' + y = e^x$$

Try e^x:

$$\text{LHS}: \ e^x - 2e^x + e^x = 0$$

Try xe^x:

$$\text{LHS}: \ xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0$$

- What next?
- If at first you don't suceed ... multiply by x again

Try x^2e^x:

$$\text{LHS}: \ x^2e^x + 4xe^x + 2e^x - 2x^2e^x - 4xe^x + x^2e^x = 2e^x$$

Hence: $\frac{1}{2}x^2e^x$ is a solution.

Examples

$$y'' + 2y' + y = x$$

Try x:

$$\text{LHS}: \ 0 + 2 + x = 2 + x$$

Try $x - 2$:

$$\text{LHS}: \ 0 + 2 + x - 2 = x$$

Solution: $x - 2$

Intelligent Guesswork

- **Iterative** method
- Start with simple functions
- Use functions that **don't** contribute much to help:
  1. Low degree polynomials, a, x, x^2
  2. Solutions of homogeneous ODE

Summary

- Wronskian:

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

- Use Wronskian to test for linear independence
- Inhomogenous ODEs:

$$y'' + p(x)y' + q(x)y = r(x)$$

- Solve by intelligent guesswork