TMA4115 Matematikk 3

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Lecture 5: The Grand Tour

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Recap

- Studying ODEs of form:

$$y'' + p(x)y' + q(x)y = 0$$

- Reduction of order: use one solution to get another
- Constant coefficients: exponential solutions

Key Points

- Another type of ODE: Euler–Cauchy
- Idea of uniqueness
- Test for linear independence
- (Introducing: the Wronskian!)

ODEs and Polynomials

Recall
To solve

$$y'' + ay' + by = 0$$

consider

$$\lambda^2 + a\lambda + b$$

(**auxilliary** or **characteristic** equation)

$$\text{ODEs} \longleftrightarrow \text{Polynomials}$$

Not a coincidence (learn more in *Fourier analysis*).

Euler–Cauchy ODEs

Introducing . . .
The Euler–Cauchy equations:

$$x^2 y'' + axy' + by = 0$$

Standard Form:

$$y'' + ax^{-1}y' + bx^{-2}y = 0$$

- Why? Because we can solve it.
- Anticipate "issues" at 0!
- Similar to $y'' + ay' + by = 0$

How to Solve an ODE: Guess!

- Degree shift: try x^m
- Substitute in:

$$m(m-1)x^{m-2} + ax^{-1}mx^{m-1} + bx^{-2}x^m = 0$$

- Gather terms:

$$\big(m(m-1) + ma + b\big)x^{m-2} = 0$$

- Auxilliary equation:

$$m^2 + (a-1)m + b$$

Solution

Solution of Cauchy–Euler
If m is a root of

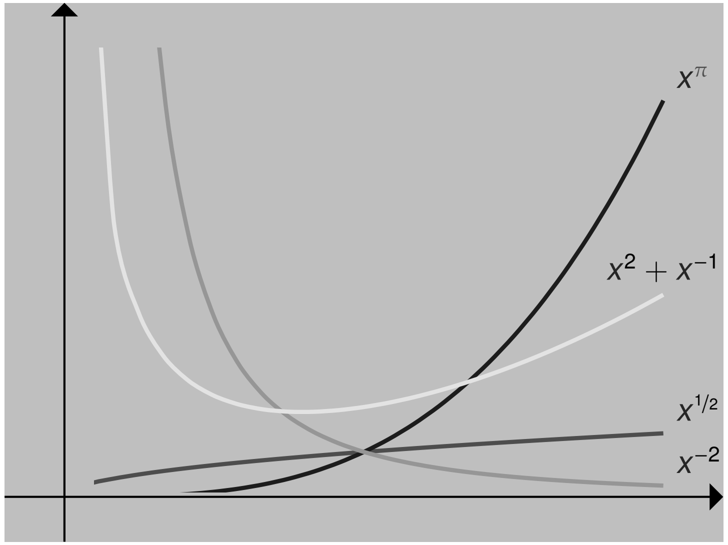
$$m^2 + (a-1)m + b$$

then x^m satisfies

$$x^2 y'' + axy' + by = 0$$

- Note: $m \in \mathbb{R}$. *Not* $\mathbb{Z}$*!*
- 2 distinct real roots $\Longrightarrow$ 2 distinct solutions
- Potential problems at $x = 0$ (as expected)
- 1 real root or 2 complex roots not much harder
  But not on syllabus. . .

Pretty Pictures



Spot the Difference

Compare and Contrast

$$x^2 y'' + axy' + by = 0$$
$$y'' + ax^{-1}y' + bx^{-2}y = 0$$

Case Study

$$x^2 y'' - 3xy' + 3y = 0$$

Auxilliary equation:

$$m^2 + (-3 - 1)m + 3$$

Roots:

$$m = 1, \quad m = 3$$

Solutions:

$$cx + dx^3$$

So far so good

Case Study

$$x^2 y'' - 3xy' + 3y = 0, \qquad cx + dx^3$$

Initial conditions: $y(0) = 0$, $y'(0) = 0$.

$$c0 + d0 = 0$$
$$c + d0 = 0$$

Conclusion: Infinitely many solutions!

Initial conditions: $y(0) = 1$, $y'(0) = 0$.

$$c0 + d0 = 1$$
$$c + d0 = 0$$

Conclusion: No solutions!

Caveat Solvator

Important

Functions make sense at $x = 0$
$$\not\Longrightarrow$$
Solutions make sense at $x = 0$

Spot The Difference

Compare and Contrast

$$x^2 y'' + axy' + by = 0$$
$$y'' + ax^{-1}y' + bx^{-2}y = 0$$

Solution

- No **substantial** difference
- Standard form **clearer** hence **better**

What Is an ODE?

Question
What is an ODE?

Answer
A way of **specifying** a curve by its derivatives.

Key word: specifying.
If it doesn't **specify**, it isn't (so) useful!

Uniqueness or Existence

Theorem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(x_0) = K_0, y'(x_0) = K_1$$

Interval I:

- $p(x)$, $q(x)$ *continuous on* I
- $x_0 \in I$.

Then IVP has a solution and it is unique.

Existence There is a solution

Uniqueness The solution is unique

Uniqueness By Squeezing

Key Property of Continuous Functions

$$f\colon [a,b] \to \mathbb{R} \text{ continuous}$$
$$\exists t_0 \in [a,b] : |f(t_0)| \geq |f(t)| \forall t \in [a,b]$$
i.e. $|f|$ achieves its maximum

Apply to: $p(x)$, $q(x)$, $y'(x)$.

Simplifying Assumptions:

1. Work on $[0,1]$
2. $|p(x)|, |q(x)| \leq 1/4$ all $x \in [0,1]$
3. Consider IVP with $y(0) = 0$, $y'(0) = 0$

Know that $y(x) = 0$ is a solution.

Goal: Show it is the **only** solution.

Uniqueness By Squeezing

$$y'' + p(x)y' + q(x)y = 0, \quad y(0) = 0,\ y'(0) = 0$$

Assume: $y\colon [0,1] \to \mathbb{R}$ is a solution.

By **Key Property of Continuous Functions**:
there is $x_0 \in [0,1] : |y'(x_0)| \geq |y'(x)|$ all $0 \leq x \leq 1$

Integration $\implies |y(x)| \leq |y'(x_0)|$ also

Uniqueness By Squeezing

Then:

$$\begin{aligned}
y'(x_0) &= \int_0^{x_0} y''(x)\mathrm{d}x \\
&= -\int_0^{x_0} p(x)y'(x) + q(x)y(x)\mathrm{d}x \\
|y'(x_0)| &\leq \int_0^{x_0} |p(x)||y'(x)| + |q(x)||y(x)|\mathrm{d}x \\
&\leq \frac{1}{2}\int_0^{x_0} |y'(x_0)|\mathrm{d}x \\
&\leq \frac{1}{2}|y'(x_0)|x_0 \leq \frac{1}{2}|y'(x_0)|
\end{aligned}$$

Only possible if $|y'(x_0)| = 0$

But $|y'(x)| \leq |y'(x_0)|$ so $y'(x) = 0$ **for all** $x \in [0,1]$!

So $y(x) = 0$ and solution is unique.

Uniqueness Facts

$$y'' + p(x)y' + q(x)y = 0$$

1. $y_1 \neq 0$ solution
   $\implies y_1(x)$ and $y_1'(x)$ never **simultaneously** zero
   If so, y_1 solution with $y(x_0) = 0, y'(x_0) = 0$
   Uniqueness $\implies y_1 = 0$
2. $y_1 \neq 0$, y_2 solutions
   $y_1(x_0) = 0 = y_2(x_0)$ some x_0
   $\implies y_2 = ky_1$ some $k \in \mathbb{R}$
   Non-zero $\implies y_1'(x_0) \neq 0$
   $\implies \frac{y_2'(x_0)}{y_1'(x_0)} y_1 + y_2$ satisfies **??**
3. Same with $y_1'(x_0) = 0 = y_2'(x_0)$.

Uniqueness Example

Example

$$y'' + 4y = 0$$

Solutions: $\sin(2x)$, $\cos(2x)$, $\sin(x)\cos(x)$.

1. $\sin(2x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\sin(2x)$ never simultaneously zero
2. $\sin(2\pi) = 0 = \sin(\pi)\cos(\pi) \Longrightarrow \sin(2x) = 2\sin(x)\cos(x)$

Linear Independence

Uniqueness motivates linear independence:

Theorem

y_1, y_2 *linearly independent solutions*

$$\Longleftrightarrow$$

All *solutions* ***uniquely*** *of form*

$$ay_1 + by_2$$

Recall:

Definition

y_1, y_2 *linearly independent*

$$\Longleftrightarrow$$

$$ay_1 + by_2 = 0 \Longrightarrow a = b = 0$$

Problem: Have to test **all** a, b and **all** x

Testing Linear Independence

y_1, y_2 not linearly independent

$$\Longleftrightarrow$$

one is a multiple of the other

$$\Longleftrightarrow$$

$y_2 = ky_1$ or $y_1 = 0 \Longleftrightarrow$

$y_2(x) = ky_1(x)\forall x$ or $y_1(x) = 0\forall x$

Problem: k unknown *if it exists*
Solution: use a x_0 to limit choices

Testing Linear Independence

$\big(y_1(x_0) = 0$ and $y_1'(x_0) = 0\big) \Longrightarrow \big(y_1 = 0\big) \Longrightarrow$ dependent.
So if $y_1 \neq 0$, at least one of

$$\frac{y_2(x_0)}{y_1(x_0)}, \qquad \frac{y_2'(x_0)}{y_1'(x_0)}$$

is well-defined and **if** $y_2 = ky_1$ then k is one of them.

Tests

1. Is $y_1 = 0$?
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)}y_1$?
3. Is $y_1'(x_0) \neq 0$ and $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)}y_1$?

Yes to any $\Longrightarrow$ dependent **No to all** $\Longrightarrow$ independent

Testing Linear Independence

1. Is $y_1 = 0$?
   Are $y_1(x_0) = 0$ and $y'_1(x_0) = 0$?
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
   $y_3 := y_2 - \frac{y_2(x_0)}{y_1(x_0)} y_1$ solution with $y_3(x_0) = 0$
   $y_3 = 0 \iff y'_3(x_0) = 0$
   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y'_2(x_0) = \frac{y_2(x_0)}{y_1(x_0)} y'_1(x_0)$
   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y'_2(x_0) y_1(x_0) - y_2(x_0) y'_1(x_0) = 0$
3. Is $y'_1(x_0) \neq 0$ and $y_2 = \frac{y'_2(x_0)}{y'_1(x_0)} y_1$?
   $y_3 := y_2 - \frac{y'_2(x_0)}{y'_1(x_0)} y_1$ solution with $y'_3(x_0) = 0$
   $y_3 = 0 \iff y_3(x_0) = 0$
   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y_2(x_0) = \frac{y'_2(x_0)}{y'_1(x_0)} y_1(x_0)$
   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y_2(x_0) y'_1(x_0) - y'_2(x_0) y_1(x_0) = 0$

Testing Linear Independence

Claim

y_1 and y_2 linearly independent
$\iff$
$y_1(x_0) y'_2(x_0) - y'_1(x_0) y_2(x_0) = 0$
for **some** x_0

How can it be zero?

1. $y_1(x_0) = 0$ and $y'_1(x_0) = 0 \Longrightarrow$ test 1 holds
2. $y_1(x_0) \neq 0 \Longrightarrow$ test 2 holds
3. $y_1(x_0) \neq 0 \Longrightarrow$ test 3 holds

Introducing . . . The Wronskian

Definition

*The **Wronskian** of two functions*
y_1, y_2
is

$$W(y_1, y_2) := y_1 y'_2 - y'_1 y_2$$

Examples

1. $W(x, x^2) = x 2x - x^2 = x^2$
2. $W(e^x, e^{-x}) = e^x(-e^{-x}) - e^x e^{-x} = -2$
3. $W(\cos(x), \sin(x)) = \cos(x)\cos(x) - (-\sin(x)\sin(x)) = 1$

Wronskian and Linear Independence

$$y'' + p(x)y' + q(x)y = 0$$

y_1 and y_2 solutions of ODE.
y_1, y_2 linearly independent
$\iff$
$W(y_1, y_2)(x) = 0$ for all x
$\iff$
$W(y_1, y_2)(x_0) = 0$ for some x_0

The Norwegian Connection

Really Weird Fact

$W(y_1, y_2)$ depends on ODE and **almost** not on y_1 and y_2!

$$y'' - 2y' + 2y = 0$$

Solutions: $ax + bx^2$

$$\begin{aligned} W(a_1x + b_1x^2, a_2x + b_2x^2) &= (a_1x + b_1x^2)(a_2 + 2b_2x) \\ &\quad - (a_1 + 2b_1x)(a_2x + b_2x^2) \\ &= a_1a_2x + (2a_1b_2 + a_2b_1)x^2 + 2b_1b_2x^3 \\ &\quad - a_1a_2x - (a_1b_2 + 2a_2b_1)x^2 - 2b_1b_2x^3 \\ &= (a_1b_2 - a_1b_1)x^2 \end{aligned}$$

Always a multiple of x^2!

Abel's Identity

$$\begin{aligned} W(y_1, y_2)' &= y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' \\ &= y_1(-py_2' - qy_2) - (-py_1' - qy_1)y_2 \\ &= -p(y_1y_2' - y_1'y_2) \\ &= -pW(y_1, y_2) \end{aligned}$$

1st Order Linear ODE!
Solution:

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)e^{-\int_{x_0}^{x} p(t)\mathrm{d}t}$$

Curiousity for now, useful later.

Summary

- Euler–Cauchy ODEs provide useful examples of **solvable** ODEs
- Uniqueness is **extremely** useful
- Standard form important to see extent of ODE
- Linear independence testable by computing

  $$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)$$

  at **any** x_0 in the interval of interest
- Above formula will be useful, so give it a name: **Wronskian**