TMA4115 Matematikk 3

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Lecture 3: Two ODEs Are Better Than One

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Key Ideas

- ODEs occur everywhere
- Solving ODEs is an art
- “Ends justify the means”

Stories

A Mathematical Topic Has. . .

1. Story
2. Techniques
3. Examples

Differential Equations: The Story

The Good News

nice Differential Equation
+
precise initial conditions
=
unique solution.

- "Nice": everything twice differentiable is enough
- "Precise": *n* bits of information for *n*th order

The Bad News

No method to **find** solution!

Solving ODEs: An Analogy

Solve:

$$\frac{dy}{dx} = \sqrt{1+y^2}$$

Separation of variables:

$$\int \frac{dy}{\sqrt{1+y^2}} = \int dx$$

Solution:

$$y(x) = \sinh(x)$$

Solving ODES $\longleftrightarrow$ Integration

Differentation is an **algorithm**
Integration is an **art**

Solving ODEs

The Problem

Chance of random ODE having exact solution:
million to 1

The Solution

Chance of needing to solve ODE with exact solution:
9 out of 10

"Exact solution" really means "well-studied solution"!

Finding Solutions

Epilogue

However you find a solution **it is the solution!**

- No "silver bullet" $\Longrightarrow$ lots of "steel bullets".
- No **one** technique will solve all ODEs.
- Techniques can involve ... dubious ... steps.
- **But that doesn't matter!**
- Technique = informed guesswork.

Types of ODEs

No single technique $\Longrightarrow$ sort ODEs by technique

Introducing . . .

2nd Order Linear Ordinary Differential Equations

2nd Order

Has y'' but not y'''

Linear

Has y but not y^2.

Example

$$y'' + \sin(x)y' + \cos(x)y = \tan(x)$$
$$y''y' + y^2 = \sin(x)$$

Why?

Why Linear?

- Most ODEs "in nature" are linear
- Linear $\Longrightarrow$ Decomposable $\Longrightarrow$ Simple

(Not independent statements)

Decomposable

$a, b \in \mathbb{R}$,
$y'' + \sin(x)y' + \cos(x)y = 0$
has unique solution $y_{a,b}(x)$ with
$y_{a,b}(0) = a$, $y'_{a,b}(0) = b$
Decomposable $\Longrightarrow y_{a,b}(x) = ay_{1,0} + by_{0,1}$

Non-Decomposable

Need to find $y_{a,b}$ for **all** $a, b \in \mathbb{R}$

Decomposable

Need to find $y_{a,b}$ for **only two** $a, b \in \mathbb{R}$

Homogeneous vs Inhomogeneous

Non-Linear

$$y''y' + y^2 = 0$$
No decomposition formula

Linear Inhomogeneous

$$y'' + \sin(x)y' + \cos(x)y = \tan(x)$$
Decomposition formula, but not simplest

Linear Homogeneous

$$y'' + \sin(x)y' + \cos(x)y = 0$$
Simple decomposition formula

The Problem

Given

$$y'' + p(x)y' + q(x)y = 0$$

find y_1, y_2 such that:

1. y_1, y_2 satisfy ODE
2. If y satisfies ODE then $y = ay_1 + by_2$, $a, b \in \mathbb{R}$
   say: y is a **linear combination** of y_1 and y_2

Can replace **??** by:

??'. y_1 and y_2 are **linearly independent**:

$$ay_1 + by_2 = 0 \Longrightarrow a = b = 0$$

Neither is a **constant** multiple of the other.

An Actual Technique

Analogy
Solve:

$$x^2 + 4x - 12 = 0$$

Easy guess: 2 works.

Now **use** this to find other solution!

$$\frac{x^2 + 4x - 12 = 0}{x - 2} = x + 6$$

So -6 is other solution.
(Note: slightly dubious "divide by zero" step **didn't matter**!)

Reduction Of Order

Process

1. Guess a solution
2. Use to reduce to 1st order

Note: 1st order $\Longrightarrow$ easy!

Example

Solve:

$$y'' + y = 0$$

Easy guess: $\sin(x)$

Other solution: . . . ???
Put $\sin(x)$ into equation **somehow** to get rid of y''.
Assume can write $y = u\sin(x)$.

$$\begin{aligned} y' &= u'\sin(x) + u\cos(x) \\ y'' &= u''\sin(x) + 2u'\cos(x) - u\sin(x) \\ 0 = y'' + y &= u''\sin(x) + 2u'\cos(x) \end{aligned}$$

Example ctd

Obtain:

$$u''\sin(x) + 2u'\cos(x) = 0$$

Still 2nd Order Linear ODE.

But . . . It's actually 1st Order Linear ODE in u' !

$$U' + 2U\cot(x) = 0$$

Solution:

$$U = \mathrm{e}^{-2\int \cot(t)\mathrm{d}t} = \frac{1}{\sin^2(x)}$$

So:

$$u = \int U\mathrm{d}t = \cot(x)$$

Hence:

$$y = \cot(x)\sin(x) = \cos(x)$$

Reduction of Order

Key Idea

Use guessed solution to change:

2nd Order ODE $\longrightarrow$ Two 1st Order ODEs

Use y_1 to turn

$$y'' + p(x)y' + q(x) = 0$$

into

$$U' + \left(\frac{2y_1'}{y_1} + p\right)U = 0$$
$$u' = U$$

where $y_2 = uy_1$

A Final Interval

Solutions are often only defined on an **interval**.

Intervals

$$\mathbb{R}$$
$$(a, \infty)$$
$$[a, \infty)$$
$$(-\infty, a)$$
$$(-\infty, a]$$
$$(a, b)$$
$$[a, b)$$
$$(a, b]$$
$$[a, b]$$

Interval often obvious, **must** contain "initial position".

Summary

- 2nd order homogeneous linear ODEs: simple to solve
- 2 solutions to construct rest
- "Reduction of order": use one solution to find another
- **Finding** solution is more important than method