TMA4115 Matematikk 3

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Lecture 2: Mathematicians are $-e^{\pi i}$!

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Recap

- Complex numbers $\mathbb{C}$
- $z = x + iy$
  $x = \operatorname{Re} z$, $y = \operatorname{Im} z$
- Add, Subtract, Multiply, Divide
  just as in $\mathbb{R}$
- Complex conjugation: $\overline{x + iy} = x - iy$

Key Points

- Making sense of z^w
- Geometry of $\mathbb{C}$

The Power of Powers

Why it might work

Multiplication is repeated addition and $\times$ extends
Powers are repeated multiplication

Why it might not work

Taking powers is not **symmetric**
Powers in $\mathbb{R}$ already have problems

Problems with Powers

Non-Symmetry

$$a + b = b + a$$
$$a \times b = b \times a$$
$$a^b \neq b^a$$

- a and b play **different** roles
- So replacing a by z does **not necessarily** mean we can replace b by w
- E.g. n^m fine for $n \in \mathbb{Z}, m \in \mathbb{N}$ but not for $n, m \in \mathbb{Z}$

But sometimes **it just works anyway!**

Problems with Powers, pt II

Problems in $\mathbb{R}$

$$a^b$$

not defined for all $a, b \in \mathbb{R}$!
Works for:

- $a \in (0, \infty)$, $b \in \mathbb{R}$
- $a \in \mathbb{R} \setminus \{0\}$, $a \in \mathbb{Z}$
- $a \in \mathbb{R}$, $a \in \mathbb{N}$

But not necessarily unique:

$$4^{1/2} = \begin{cases} 2 \\ -2 \end{cases}$$

So we proceed with **caution**!

Safe Territory

$$z^w$$

Definitely works for

- $w \in \mathbb{N}$: repeated multiplication
- $z \neq 0$, $w \in \mathbb{Z}$: reciprocal and repeated multiplication
- $z = -1$ and $w = \frac{1}{2}$

Could now proceed bit by bit, e.g. $(-4)^{1/2} = \pm i2$
But let's ...

Carpe Jugulum!

Defining Power

How do we **define**:

$$a^b?$$

Lots of ways...
One way:

$$a^b := \mathrm{e}^{b\ln(a)}$$

Advantages

- Reduces problem to $\ln$ and powers of e.
- Shows why a^b only defined for $a > 0$: $\ln(a)$ only defined for $a > 0$

Strategy

- Define e^w for as many $w \in \mathbb{C}$ as possible
- Define $\ln(z)$ for as many $z \in \mathbb{C}$ as possible
- Define $z^w := \mathrm{e}^{w\ln(z)}$

The Exponential Function

How do we **define**:

$$\mathrm{e}^b?$$

Lots of ways...
One way:

$$\text{Solve } f' = bf, \quad f(0) = 1$$

Solution: $f(t) = \mathrm{e}^{bt}$
Now evaluate at $t = 1$.
Extend to $\mathbb{C}$? Try a simple case first!
To define

$$\mathrm{e}^{\mathrm{i}}$$

$$\text{Solve } f' = \mathrm{i}f, \quad f(0) = 1$$

and evaluate at $t = 1$

Solving $f' = \mathrm{i}f$

The Double Derivative Trick: If $f(t)$ satisfies $f' = \mathrm{i}f$ then it also satisfies

$$f'' = \mathrm{i}f' = \mathrm{i}\mathrm{i}f = -f$$

Solutions:

$$f(t) = c\sin(t) + d\cos(t)$$

Key Step: $c, d \in \mathbb{C}$!

- $f(0) = 1 \Longrightarrow d = 1$
- $f'(0) = \mathrm{i}f(0) = \mathrm{i} \Longrightarrow c = \mathrm{i}$

Solution:

$$f(t) = \cos(t) + \mathrm{i}\sin(t)$$

Conclusion

So **define**

$$\mathrm{e}^{\mathrm{i}t} = \cos(t) + \mathrm{i}\sin(t)$$

In particular:

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\pi} &= \cos(\pi) + \mathrm{i}\sin(\pi) &&= -1\\
\mathrm{e}^{2\mathrm{i}\pi} &= \cos(2\pi) + \mathrm{i}\sin(2\pi) &&= 1\\
\mathrm{e}^{\mathrm{i}\pi/2} &= \cos(\pi/2) + \mathrm{i}\sin(\pi/2) &&= \mathrm{i}\\
\mathrm{e}^{\mathrm{i}} &= \cos(1) + \mathrm{i}\sin(1) &&= 0.54030 + \mathrm{i}0.84147
\end{aligned}$$

Warning: use **radians**

What About The Rest?

Easy! Define:

$$e^z = e^{x+iy} = e^x e^{iy} := e^x\Big(\cos(y) + i\sin(y)\Big)$$

So

$$e^{2+i3} = e^2\Big(\cos(3) + i\sin(3)\Big) \qquad = -7.3151 + i1.0427$$
$$e^{\pi+ie} = -21.098 + i9.5058$$

Coming up . . .

Logarithms!

After the commercial geometric break.

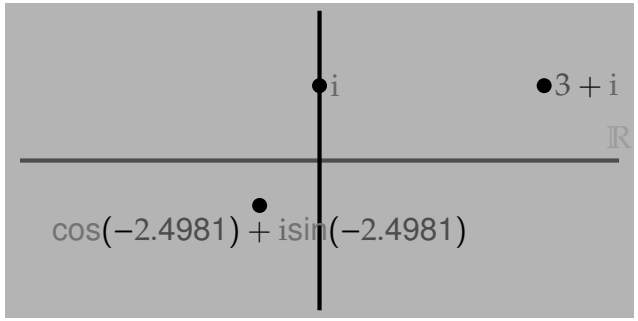
Geometry of $\mathbb{C}$

$$z = x + iy, \qquad e^{it} = \cos(t) + i\sin(t)$$

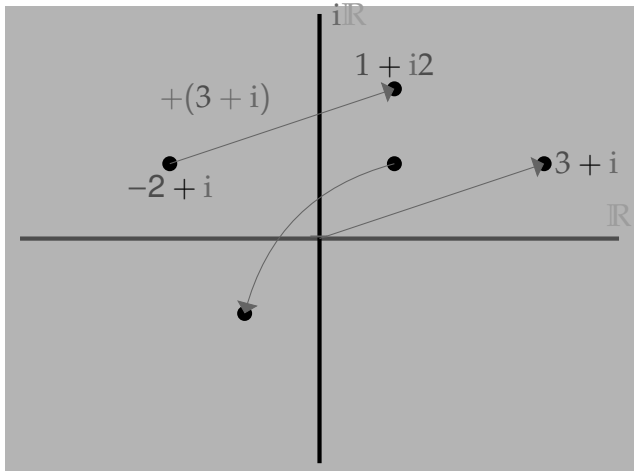
Looks a bit like . . .

Co-ordinates!

(Cartesian and polar)

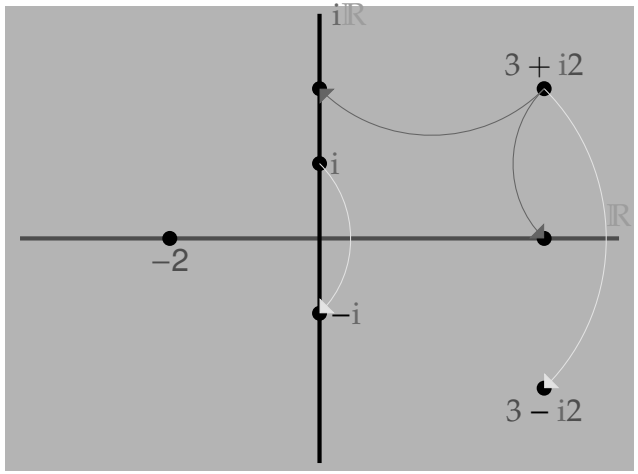


Pretty Pictures



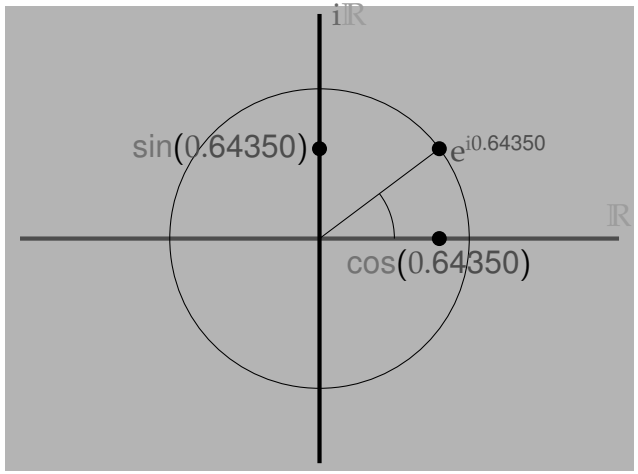
Addition, Subtraction

Pretty Pictures



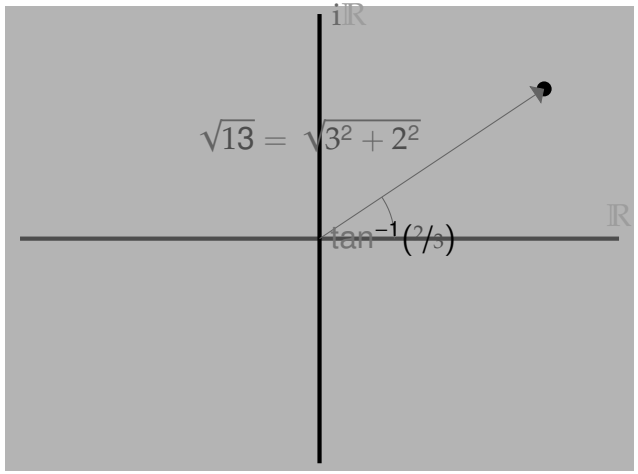
Complex conjugation, real and imaginary parts

Pretty Pictures



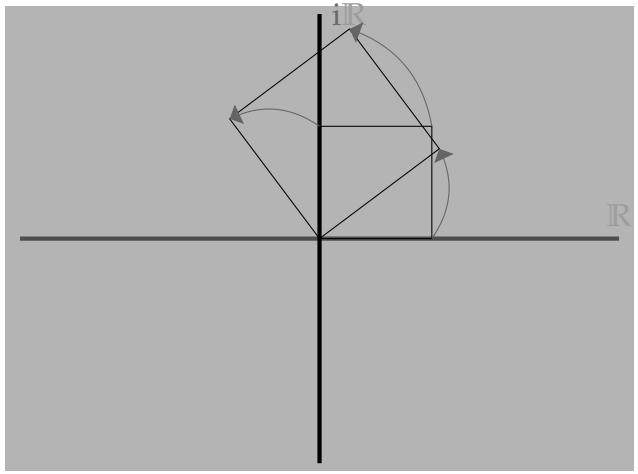
Polar Co-ordinates

Pretty Pictures



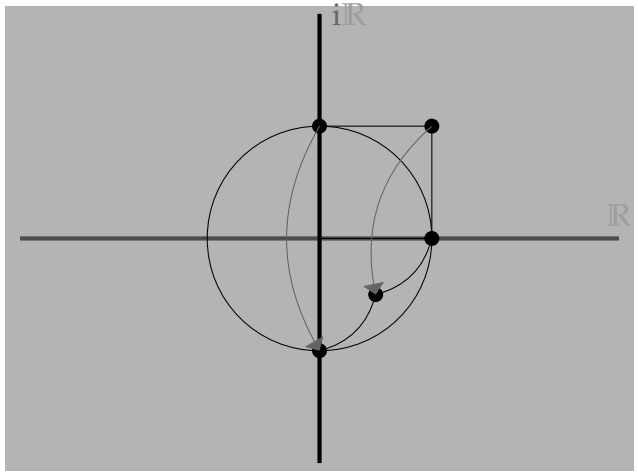
Absolute Value: $z = x + iy \mapsto |z| = \sqrt{x^2 + y^2}$
Argument: $z = x + iy \mapsto \arg(z) = \tan^{-1}(y/x)$

Pretty Pictures



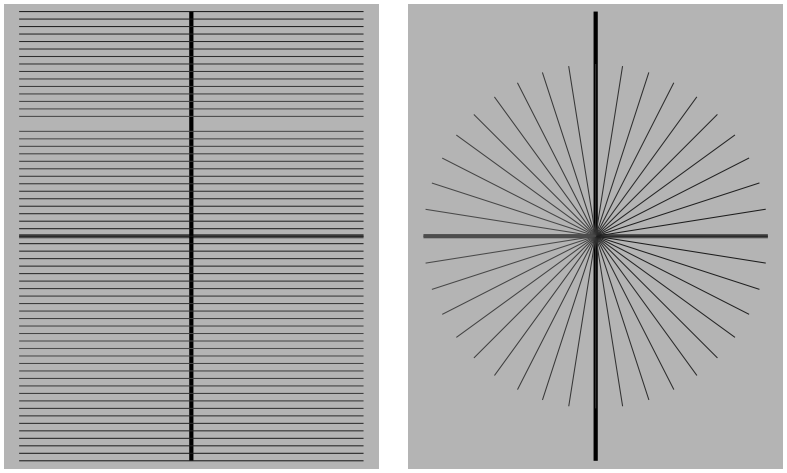
Multiplication = Rotation and Dilation

Not Just Pretty Pictures



Reciprocal: $z \mapsto \frac{1}{z}$

Not Just Pretty Pictures



$z \mapsto e^z$

Back To Logarithms

$$e^z = e^{x+iy} = e^x e^{iy} = e^x\Big(\cos(y) + i\sin(y)\Big)$$

Logarithm is inverse to exponential:

$$\ln(w) = z \iff e^z = w$$

Really easy in **polar** co-ordinates:

$$\begin{aligned} w = re^{i\theta} &= e^x e^{iy} \\ \implies \quad e^x = r, &\qquad e^{iy} = e^{i\theta} \\ \implies \quad x = \ln(r), &\qquad y = \theta + 2\pi n, \quad n \in \mathbb{Z} \end{aligned}$$

Arguments are ambiguous!

$$\ln(w) = \ln(r) + i(\theta + 2\pi n)$$

Powers

General Formula

$$z^w = e^{w\ln(z)}$$

Can be **multi-valued**!

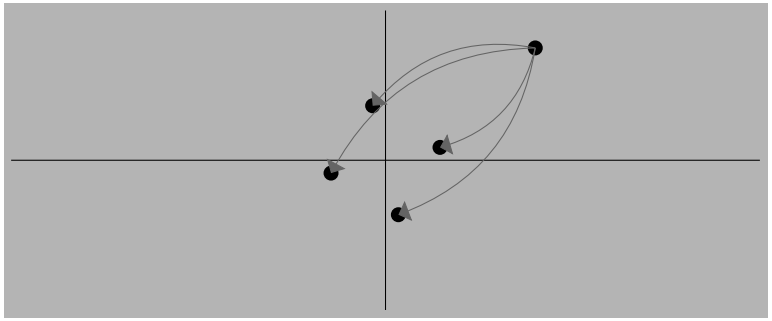
Special Case: Roots

$$\begin{aligned} z^{1/m} &= e^{\ln(z)/m} \\ &= e^{(\ln(r)+i(\theta+2\pi n))/m} \\ &= e^{\ln(r)/m + i\theta/m + i2\pi n/m} \\ &= r^{1/m} e^{i\theta/m} e^{i2\pi n/m} \\ &= r^{1/m}\Big(\cos(\theta/m + 2\pi n/m) + i\sin(\theta/m + 2\pi n/m)\Big) \end{aligned}$$

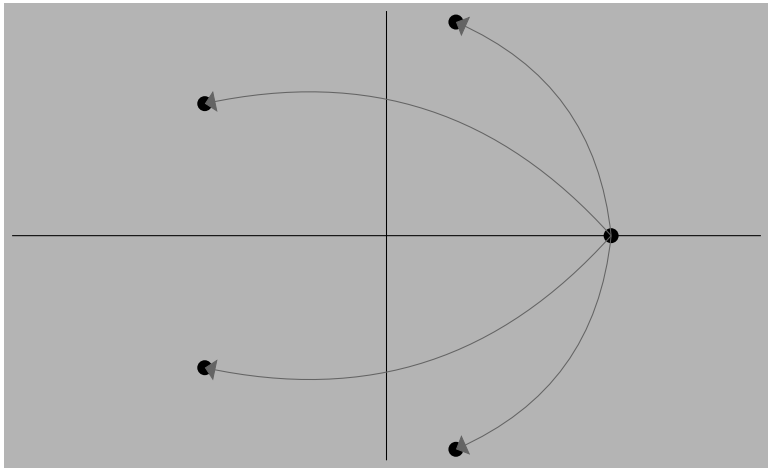
Note: Only m distinct values.

Example: 4th Roots of $4 + i3$

$$\begin{aligned} r &= \sqrt{4^2 + 3^2} = 5, & \theta &= \tan^{-1}(4/3) = 0.9273 \\ &\mapsto 5^{1/4} = 1.4953, & &\mapsto 0.23182 + n\pi/2 \end{aligned}$$



5th Roots of 1



Summary

- The **complex plane** lets us see what's happening
- Can use knowledge of $\mathbb{R}^2$ to help understand $\mathbb{C}$
- e^z makes sense
- $\ln(z)$ makes sense — multi-valued!
- $z^{1/n}$ makes sense — multi-valued!
- These have nice **geometric** pictures.