TMA4115 Matematikk 3

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Lecture 1: y i?

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Introducing . . .

$$\sqrt{-1}$$

Usual Questions

Questions

What? Square root of -1

Why? Because it's useful

How? Ah, now **that's** an interesting question . . .

More detail on "why"

Mathematics is a **language**:

- Words $\longleftrightarrow$ Definitions
- Sentences $\longleftrightarrow$ Theorems

(Roughly)

Need **words** to express **ideas**
Words have no **intrinsic** reality;
their worth comes from their **usefulness**.

- Does "dog" exist?
- Does "$\sqrt{-1}$" exist?

Wrong Question!

Is $\sqrt{-1}$ Useful?

Real Question
What **concepts** does $\sqrt{-1}$ allow us to talk about?

- Roots of any polynomial, e.g. $x^2 + 1$
  Seems boring, but **incredibly** useful
- Signal analysis
- Electromagnetism
- Quantum Theory
- ...

The shortest path between two truths in the real domain passes through the complex plane.
Hadamard

But How?

Complex Numbers
$=$
Real Numbers
$+$
$\sqrt{-1}$

Is that it?

Of course not!

Form Versus Function

Slogan
Mathematical objects are determined more by what they **do** than what they **are**.

Dynamic not static.

What can we do with numbers?

1. Add: $2 + 3 = 5$, $\pi + e = 5.8599\ldots$
2. Subtract: $3 - 2 = 1$, $\pi - e = 0.42331\ldots$
3. Multiply: $2 \times 3 = 6$, $\pi \times e = 8.5397\ldots$
4. Divide (if non-zero): $6 \div 3 = 2$, $\pi \div e = 1.1557\ldots$
5. Compare: $2 \leq 3$, $\pi \geq e$
6. Subject to (lots of) rules.

We'd like to do these with complex numbers.

Notational Pause

Warning
Notation $\sqrt{-1}$ is **unsafe**!

Problem
Already in $\mathbb{R}$, square roots are not unique.

Solution
Don't add $\sqrt{-1}$.
Add i and add the **rule** $\mathrm{i}^2 = -1$.

Set of Complex Numbers
Written as $\mathbb{C}$.

Generating Complex Numbers

- Start: $\mathbb{R} \subseteq \mathbb{C}$ and $\mathrm{i} \in \mathbb{C}$.
- Add: $3 + \mathrm{i} \in \mathbb{C}$, $x + \mathrm{i} \in \mathbb{C}$,
- Multiply: $\mathrm{i}2 \in \mathbb{C}$, $\mathrm{i}y \in \mathbb{C}$,
- Back to Add: $3 + \mathrm{i}2 \in \mathbb{C}$, $x + \mathrm{i}y \in \mathbb{C}$.

Story So Far

$$x, y \in \mathbb{R} \Longrightarrow x + \mathrm{i}y \in \mathbb{C}$$

Examples

1. $2 + \mathrm{i}3$
2. $1 + \mathrm{i}(-1)$
3. $\pi + \mathrm{i}\mathrm{e}$

What Else?

Question
What else must we have?
(So that $+$, $-$, $\times$, $\div$ defined)

Answer
Nothing!
Because of the **rules**.

Rules? What rules?
Same as $\mathbb{R}$.

Addition

$$\begin{aligned}(\mathrm{e} + \mathrm{i}3) + (\pi + \mathrm{i}2) &= (\mathrm{e} + \mathrm{i}3) + (\mathrm{i}2 + \pi)\\ &= \mathrm{e} + (\mathrm{i}3 + (\mathrm{i}2 + \pi))\\ &= \mathrm{e} + ((\mathrm{i}3 + \mathrm{i}2) + \pi)\\ &= \mathrm{e} + (\pi + \mathrm{i}(3 + 2))\\ &= (\mathrm{e} + \pi) + \mathrm{i}5\end{aligned}$$

Of the form $x + \mathrm{i}y$ with $x, y \in \mathbb{R}$.

General Rule

$$(x + \mathrm{i}y) + (u + \mathrm{i}v) = (x + u) + \mathrm{i}(y + v)$$

Subtraction

$$\begin{aligned}(\mathrm{e}+\mathrm{i}3)-(\pi+\mathrm{i}2) &= (\mathrm{e}+\mathrm{i}3)+((-\pi)+\mathrm{i}(-2))\\ &= (\mathrm{e}-\pi)+\mathrm{i}(3-2)\\ &= (\mathrm{e}-\pi)+\mathrm{i}1\\ &= (\mathrm{e}-\pi)+\mathrm{i}\end{aligned}$$

Of the form $x+\mathrm{i}y$ with $x, y \in \mathbb{R}$.

General Rule

$$(x+\mathrm{i}y)-(u+\mathrm{i}v) = (x-u)+\mathrm{i}(y-v)$$

Multiplication

$$\begin{aligned}(\mathrm{e}+\mathrm{i}3)(\pi+\mathrm{i}2) &= \mathrm{e}\pi+\mathrm{ei}2+\mathrm{i}3\pi+\mathrm{i}3\mathrm{i}2\\ &= \mathrm{e}\pi+\mathrm{ie}2+\mathrm{i}3\pi+\mathrm{i}^2 6\\ &= \mathrm{e}\pi+\mathrm{i}(\mathrm{e}2+3\pi)+\mathrm{i}^2 6\\ &= \mathrm{e}\pi+\mathrm{i}(\mathrm{e}2+3\pi)+-1\times 6\\ &= (\mathrm{e}\pi-6)+\mathrm{i}(\mathrm{e}2+3\pi)\end{aligned}$$

General Rule

$$(x+\mathrm{i}y)(u+\mathrm{i}v) = (xu-yv)+\mathrm{i}(xv+yu)$$

Division

Question

What is division?

Answer

$$\frac{1}{x}\times x = 1$$

Question

What is 1?

Answer

$$1\times x = x$$

Division

- Finding 1: solve

$$(x+\mathrm{i}y)(u+\mathrm{i}v) = (u+\mathrm{i}v)$$

- Specific instance: solve

$$(x+\mathrm{i}y)(2+\mathrm{i}3) = 2+\mathrm{i}3$$

- Expand out multiplication:

$$2x-3y+\mathrm{i}(2y+3x) = 2+\mathrm{i}3$$

- Key Fact: $u+\mathrm{i}v = s+\mathrm{i}t \iff u = s$ and $v = t$
- Solve: $2x-3y = 2$ and $2y+3x = 3$
- Solution: $x = 1$, $y = 0$.
- Conclusion: $1 = 1+\mathrm{i}0$ (unsurprisingly)

Division

- Finding $1/(x+\mathrm{i}y)$: solve

$$(x+\mathrm{i}y)(u+\mathrm{i}v) = 1+\mathrm{i}0$$

- Specific instance: solve

$$(2+\mathrm{i}3)(u+\mathrm{i}v) = 1+\mathrm{i}0$$

- Expand out multiplication:

$$2u-3v+\mathrm{i}(2v+3u) = 1+\mathrm{i}0$$

- Solve: $2u-3v=1$ and $2v+3u=0$
- Substitute: $v=-3u/2$
- Get: $u(2+9/2)=1$
- Solution: $u=2/13$, $v=-3/13$.

Division

General Formula
$x+\mathrm{i}y \neq 0$ then

$$\frac{1}{x+\mathrm{i}y} = \frac{x}{x^2+y^2} - \mathrm{i}\frac{y}{x^2+y^2}$$

Looks a bit scary ... split it up:

$$\frac{1}{x+\mathrm{i}y} = \frac{1}{x^2+y^2}(x-\mathrm{i}y)$$

Story So Far

Complex Numbers

- Notation $\mathbb{C}$
- $z \in \mathbb{C}$ represented as $z = x+\mathrm{i}y$ with $x, y \in \mathbb{R}$
- Add, Subtract, Multiply, Divide
- **Same** rules as in $\mathbb{R}$ with ...
- $\mathrm{i}^2 = -1$
- $\mathbb{R} \subseteq \mathbb{C}$ as $x \mapsto x+\mathrm{i}0$
- Conventions: leave out 1 or 0 if unambiguous:

$$2+\mathrm{i}0 \to 2, \qquad 2+\mathrm{i}1 \to 2+\mathrm{i}$$

Complex Conjugation

Division formula:

$$\frac{1}{x+\mathrm{i}y} = \frac{1}{x^2+y^2}(x-\mathrm{i}y)$$

$x-\mathrm{i}y$ is the **complex conjugate** of $x+\mathrm{i}y$
written: $\overline{x+\mathrm{i}y}$

Examples

- $\overline{2+\mathrm{i}3} = 2-\mathrm{i}3$
- $\overline{3} = \overline{3+\mathrm{i}0} = 3-\mathrm{i}0 = 3$
- $\overline{\mathrm{i}4} = -\mathrm{i}4$

Real and Imaginary Parts

$$z = x + \mathrm{i}y$$

- x is the **real part** of z

$$x = \operatorname{Re} z = \frac{1}{2}(z + \overline{z})$$

$$\operatorname{Re}(2 + \mathrm{i}3) = 2$$

- y is the **imaginary part** of z

$$y = \operatorname{Im} z = \frac{1}{2}(z - \overline{z})$$

$$\operatorname{Im}(2 + \mathrm{i}3) = 3$$

Summary

- Complex numbers are like real numbers with i
- Add, Subtract, Multiply, Divide just as in $\mathbb{R}$
- New thing: complex conjugation

Things To Ponder

- Can we compare complex numbers? $z \leq w$?
- Gaussian integers: all complex numbers of form $n + \mathrm{i}m$. How does this compare to $\mathbb{Z}$?