TMA4115 Matematikk 3

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Lecture 24: Foxes and Rabbits

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Key Points

- Eigenvectors simplify problems
- Diagonalisation is hard
- Partial information can suffice

Recap

- Basis is a “point of view”

The Pendula, Yet Again

$$\begin{bmatrix} y_a' \\ y_b' \\ z_a' \\ z_b' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 1 & -5 & 0 \\ 1 & -5 & 0 & -5 \end{bmatrix} \begin{bmatrix} y_a \\ y_b \\ z_a \\ z_b \end{bmatrix}$$

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$$w_+ = y_a + y_b + z_a + z_b$$

$$x_+ = 4y_a + 4y_b + z_a + z_b$$

$$w_- = 2y_a - 2y_b + z_a - z_b$$

$$x_- = 3y_a - 3y_b + z_a - z_b$$

$$\begin{bmatrix} w_+' \\ x_+' \\ w_-' \\ x_-' \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} w_+ \\ x_+ \\ w_- \\ x_- \end{bmatrix}$$

Key Question

Question

Where do w_+ (etc) come from?

Reconstruction

$$\begin{aligned}
w_+ &= y_a + y_b + z_a + z_b \\
x_+ &= 4y_a + 4y_b + z_a + z_b \\
w_- &= 2y_a - 2y_b + z_a - z_b \\
x_- &= 3y_a - 3y_b + z_a - z_b
\end{aligned}$$

Reconstruction

$$\begin{bmatrix} w_+ \\ x_+ \\ w_- \\ x_- \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 4 & 1 & 1 \\ 2 & -2 & 1 & -1 \\ 3 & -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_a \\ y_b \\ z_a \\ z_b \end{bmatrix}$$

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$$\begin{bmatrix} y_a \\ y_b \\ z_a \\ z_b \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -1 & 1 & -3 & 3 \\ -1 & 1 & 3 & -3 \\ 4 & -1 & 9 & -6 \\ 4 & -1 & -9 & 6 \end{bmatrix} \begin{bmatrix} w_+ \\ x_+ \\ w_- \\ x_- \end{bmatrix}$$

Differentiation

$$\begin{bmatrix} w'_+ \\ x'_+ \\ w'_- \\ x'_- \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 4 & 1 & 1 \\ 2 & -2 & 1 & -1 \\ 3 & -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} y'_a \\ y'_b \\ z'_a \\ z'_b \end{bmatrix}$$

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Eigenvectors

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Definition

$\mathbf{v}$ *is an eigenvector*
of a square matrix A *with eigenvalue* λ
if
$\mathbf{v} \neq \mathbf{0}$ *and* $A\mathbf{v} = \lambda \mathbf{v}$

Matrix ODEs: General Method

To Solve

$$\mathbf{u}'(t) = A\mathbf{u}(t)$$

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2. Solution with $\mathbf{u}(0) = \mathbf{v}_i$ is $\mathbf{u}(t) = \mathrm{e}^{-\lambda_i t}\mathbf{v}_i$
3. For general $\mathbf{v}$, try to write as $\mathbf{v} = \mu_1 \mathbf{v}_1 + \cdots + \mu_k \mathbf{v}_k$ then

$$\mathbf{u}(t) = \mu_1 \mathrm{e}^{-\lambda_1 t}\mathbf{v}_1 + \cdots + \mu_k \mathrm{e}^{-\lambda_k t}\mathbf{v}_k$$

Enough Eigenvectors

Remark

Works for all $\mathbf{v}$ if enough eigenvectors

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Need a basis of eigenvectors.

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Definition

A matrix is
diagonalisable
if there is a basis of the space
consisting of eigenvectors of the matrix

The Power of Diagonalisability

Recall: Predator–Prey

$$\begin{bmatrix} P_{\text{foxes}} \\ P_{\text{rabbits}} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 1.2 \end{bmatrix} \begin{bmatrix} P_{\text{foxes}} \\ P_{\text{rabbits}} \end{bmatrix}$$

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Question

What's the long-term prognosis?

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Answer

$$A^k \mathbf{u}_0$$

The Power of Diagonalisability

To compute $A^k \mathbf{u}_0$: easy if A is diagonalisable!

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If $A\mathbf{v} = \lambda \mathbf{v}$ then $A^k \mathbf{v} = \lambda^k \mathbf{v}$ so

$$\mathbf{u}_0 = \mu_1 \mathbf{v}_1 + \cdots + \mu_n \mathbf{v}_n$$

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Example: Predator–Prey

$$A^k \mathbf{u}_0 = \mu_1 (0.27)^k \begin{bmatrix} -0.92 \\ 0.40 \end{bmatrix} + \mu_2 (1.33)^k \begin{bmatrix} 0.31 \\ 0.95 \end{bmatrix}$$

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Easy to spot general trend: $P_{\text{foxes}} \simeq {}^{1}/_{3} P_{\text{rabbits}}$

Obvious Question

Question

When is a matrix diagonalisable?

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Nice Answer

Almost always (over $\mathbb{C}$)

Obvious Question

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Not So Nice Answer

Even if it is, it's hard to find the eigenvalues.

An Abelian Diversion

Finding Eigenvectors

Solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$ for both λ and $\mathbf{v}$.

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Definition

$$\det(\lambda I - A)$$

is the characteristic polynomial of A
degree = number of rows of A

Solving Characteristic Polynomials

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \lambda^2 - (a + d)\lambda + (ad - bc)$$

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Fundamental Problem of Algebra

Solving equations is hard!

Okay for 2×2 or 3×3, impractical for higher.

Examples

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad c(\lambda) = (\lambda - 1)^3$$

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$$\frac{1}{4}\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \qquad c(\lambda) = (\lambda - 5)^2 - 9 = \lambda^2 - 10\lambda + 16$$

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Eigenvalues: 2, 8; Eigenvectors: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Examples

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 1 & -5 & 0 \\ 1 & -5 & 0 & -5 \end{bmatrix} \qquad c(\lambda) = \lambda^4 + 10\lambda^3 + 35\lambda^2 + 50\lambda + 24$$

Examples

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 1 & -5 & 0 \\ 1 & -5 & 0 & -5 \end{bmatrix} \qquad c(\lambda) = \lambda^4 + 10\lambda^3 + 35\lambda^2 + 50\lambda + 24$$

Eigenvalues: -1, -2, -3, -4

Examples

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Eigenvalues: 1
Eigenvectors: e_1 (and non-zero multiples)

But nothing else!

More Reasonable Uses

Sensible Question

Do the foxes die out?

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Don’t actually need to know eigenvalues to solve this!

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$$A^k \mathbf{u}_0 = \mu_1 \lambda_1^k \mathbf{v}_1 + \mu_2 \lambda_2^k \mathbf{v}_2$$

More Reasonable Uses

Sensible Question

Do the foxes die out?

Key Point

Don't actually need to know eigenvalues to solve this!

$$A^k \mathbf{u}_0 = \mu_1 \lambda_1^k \mathbf{v}_1 + \mu_2 \lambda_2^k \mathbf{v}_2$$

if $|\lambda_i| < 1$, yes; otherwise, almost certainly not.

Enough is Enough

Key Point

Partial information about eigenvalues may be enough to answer the question

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Were the coupled pendula overdamped or underdamped?

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Were the coupled pendula overdamped or underdamped?

Answer

All eigenvalues real and negative so overdamped.

Inanity of Powers

Use of Diagonalisation

Simpler to compute A^k when A is diagonal.

Inanity of Powers

Use of Diagonalisation

Simpler to compute A^k when A is diagonal.

Misuse of Diagonalisation

Algorithm for computing eigenvalues works by computing A^k for large k!

Interpretation

Question

The eigenvectors are

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} \right\}$$

but the useful functions are

$$\begin{aligned} w_+ &= y_a + y_b + z_a + z_b \\ x_+ &= 4y_a + 4y_b + z_a + z_b \\ w_- &= 2y_a - 2y_b + z_a - z_b \\ x_- &= 3y_a - 3y_b + z_a - z_b \end{aligned}$$

What is a Vector, Exactly?

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

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Means: have 1 lot of $\mathbf{e}_1$ and 3 lots of $\mathbf{e}_2$.

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Means: have 1 lot of $\mathbf{e}_1$ and 3 lots of $\mathbf{e}_2$.

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$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Have 2 lots of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and -1 lot of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Rewriting Rule

Question

Bases $\{\mathbf{u}_j\}$, $\{\mathbf{v}_j\}$.

$$\mathbf{x} = \begin{cases} \mu_1 \mathbf{u}_1 + \cdots + \mu_n \mathbf{u}_n \\ \nu_1 \mathbf{v}_1 + \cdots + \nu_n \mathbf{v}_n \end{cases}$$

Know μ_j. What are the ν_i?

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Know μ_j. What are the ν_i?

Usual Answer

Write $\mathbf{u}_j = \sum \alpha_{ij} \mathbf{v}_i$ then $[\nu_i] = \left[\alpha_{ij}\right]\left[\mu_j\right]$.

Rewriting Rule

Question

Bases $\{\mathbf{u}_j\}$, $\{\mathbf{v}_j\}$.

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Usual Answer

Write $\mathbf{u}_j = \sum \alpha_{ij} \mathbf{v}_i$ then $[\nu_i] = \left[\alpha_{ij}\right]\left[\mu_j\right]$.

Remark: Very easy to get back-to-front.

Rewriting Rule

Unusual Answer

Find a row vector $\mathbf{c}$ so that $\mathbf{c}\mathbf{v}_i = \delta_{ij}$.
Then $v_j = \sum \mu_i \mathbf{c}\mathbf{u}_i$.

Rewriting Rule

Unusual Answer

Find a row vector $\mathbf{c}$ so that $\mathbf{c}\mathbf{v}_i = \gamma\delta_{ij}$.
Then $v_j = {}^{1}/_{\gamma} \sum \mu_i \mathbf{c}\mathbf{u}_i$.

Rewriting Rule

Unusual Answer

Find a **row** vector $\mathbf{c}$ so that $\mathbf{c}\mathbf{v}_i = \gamma\delta_{ij}$.
Then $v_j = {}^1/_\gamma \sum \mu_i \mathbf{c}\mathbf{u}_i$.

$$\left\{ \begin{matrix} \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} \\ \begin{bmatrix}4 & 4 & 1 & 1\end{bmatrix} \\ \begin{bmatrix}2 & -2 & 1 & -1\end{bmatrix} \\ \begin{bmatrix}3 & -3 & 1 & -1\end{bmatrix} \end{matrix} \right\} \left\{ \begin{bmatrix}-1\\-1\\4\\4\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\3\\-3\end{bmatrix}, \begin{bmatrix}1\\-1\\-2\\2\end{bmatrix} \right\}$$

Modes

The modes of the system of pendula correspond to the eigenvalues:

$$e^{-4t}\begin{bmatrix}-1\\-1\\4\\4\end{bmatrix},\quad e^{-t}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix},\quad e^{-3t}\begin{bmatrix}-1\\1\\3\\-3\end{bmatrix},\quad e^{-2t}\begin{bmatrix}1\\-1\\-2\\2\end{bmatrix}$$

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The functions $\mathbf{w}_\pm$, $x_\pm$ measure how much of each mode is in a particular solution.

Summary

- Diagonalisation reveals the “best possible point of view”
- Finding eigenvalues is hard
- Partial information is useful
- A clear head is needed to follow the back-and-forths of it all!