TMA4115 Matematikk 3

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

Spring 2010

Lecture 13: MatrixirtaM

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

24th February 2010

Key Points

- Invertible matrices are useful
- Recognising invertible matrices

Recap

- Special Processes $\longleftrightarrow$ Matrices
- Invertible process:

$$\text{Input} \longleftrightarrow \text{Output}$$

Invertible Processes

Invertible

A process is invertible if:

1. Each input is uniquely determined by its output
2. Each potential output is possible

Invertible Processes

Invertible

A process is invertible if:

1. Each input is uniquely determined by its output
2. Each potential output is possible

Remark

1. A process is invertible $\iff$ it can be reversed

Invertible Processes

Invertible

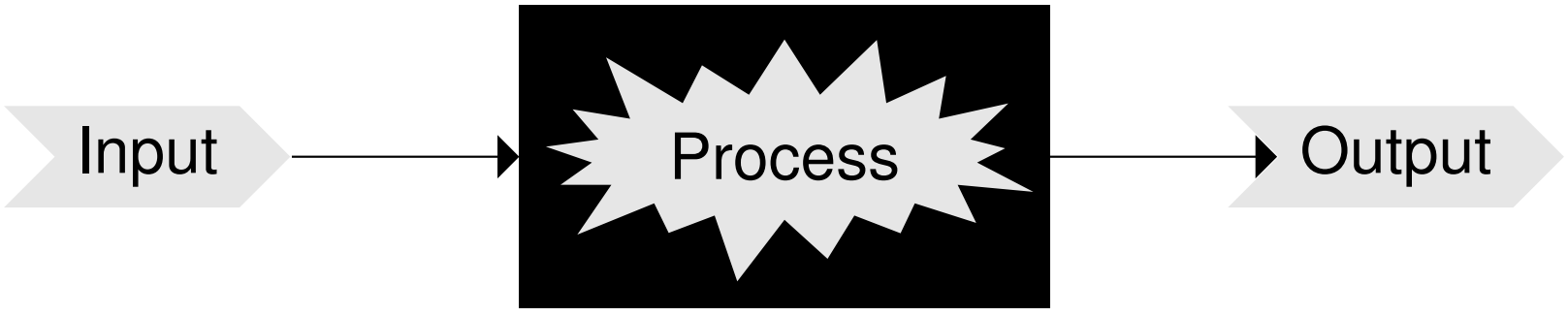
A process is invertible if:

1. Each input is uniquely determined by its output
2. Each potential output is possible

Remark

1. A process is invertible $\iff$ it can be reversed
2. There is another process which reconstructs the input from the output

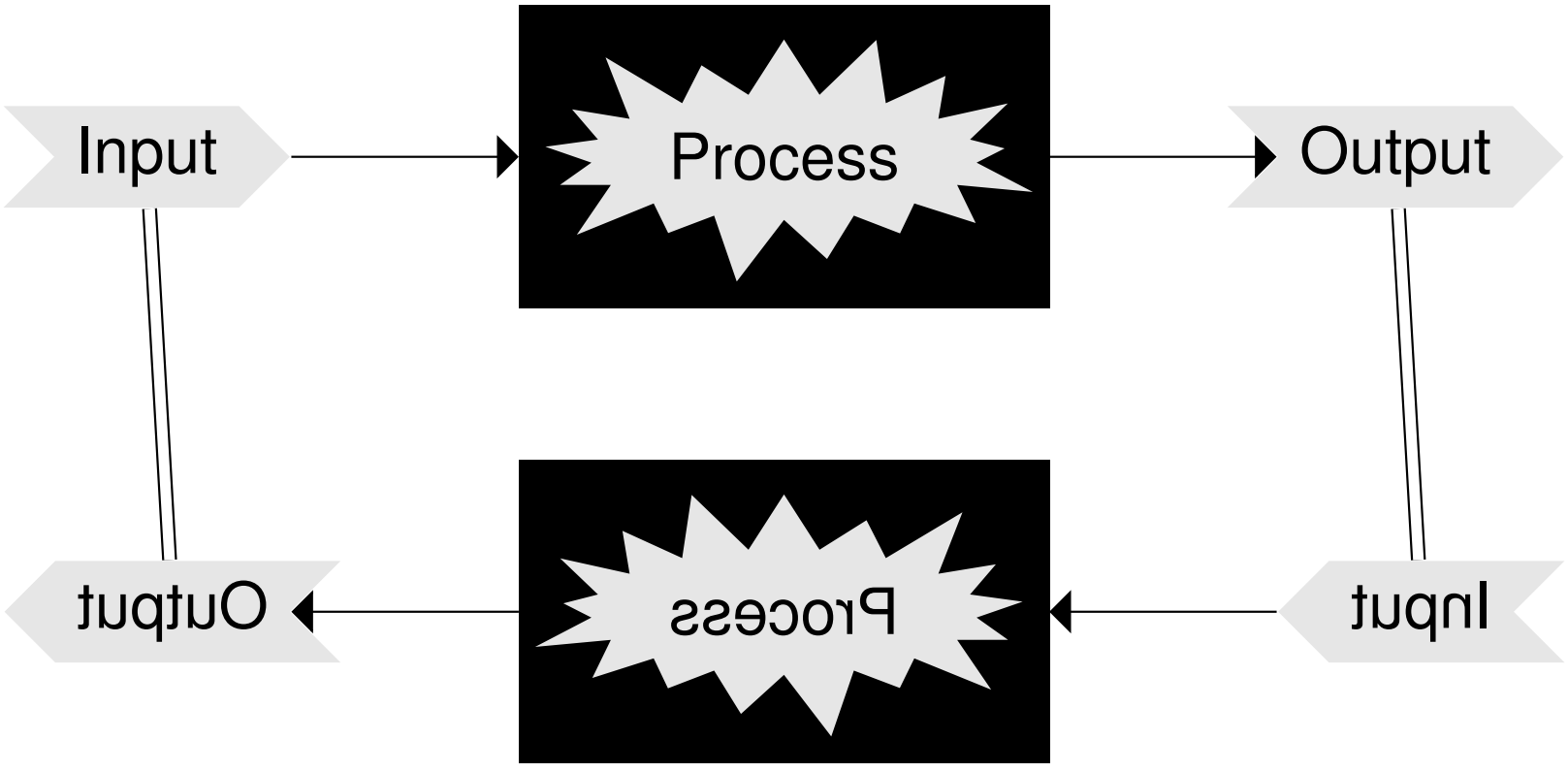
Invertible Processes



Invertible Processes

Invertible Processes

Invertible Processes



Going all the way around gets you right back where you started.

A Very Special Process

Introducing . . .

A Very Special Process

Introducing . . .

The
Do Nothing
Process!

A Very Special Process

Introducing . . .

The
Do Nothing
Process!

The “Do Nothing” process . . .

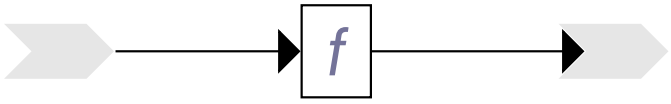
A Very Special Process

Introducing . . .

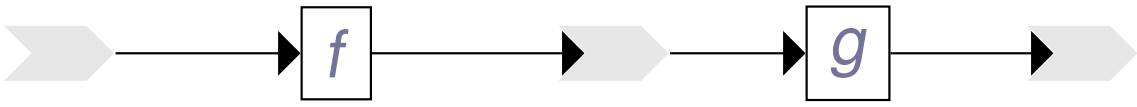
The
Do Nothing
Process!

The “Do Nothing” process . . . does nothing!

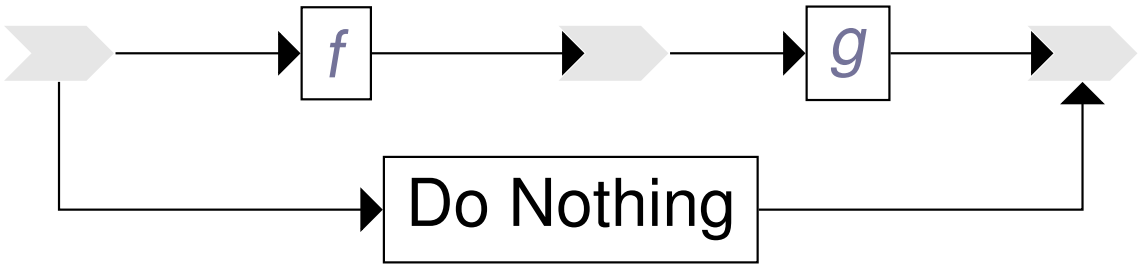
Busy Doing Nothing



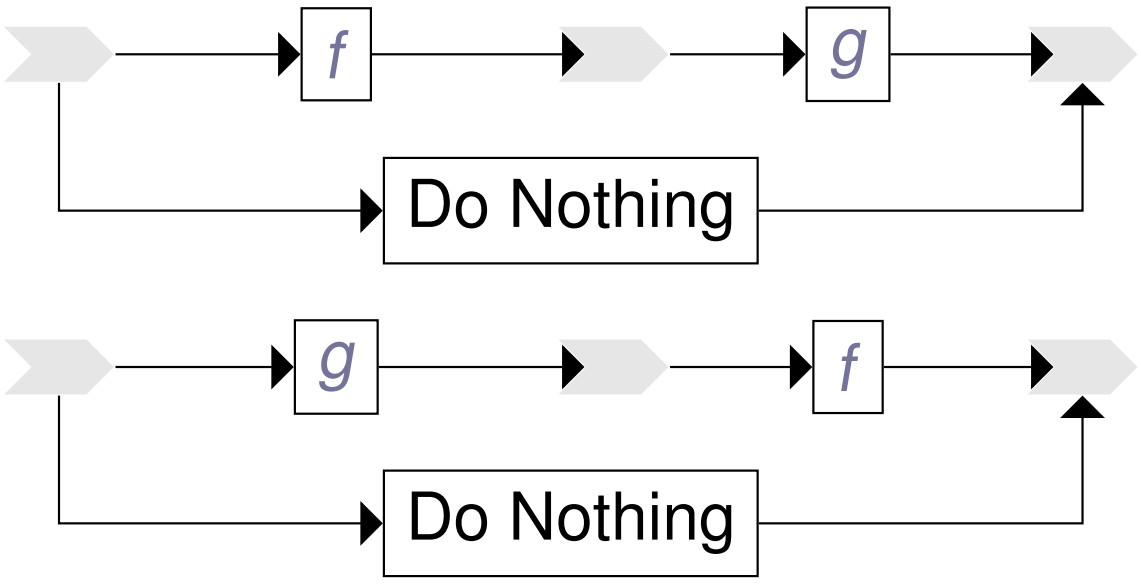
Busy Doing Nothing



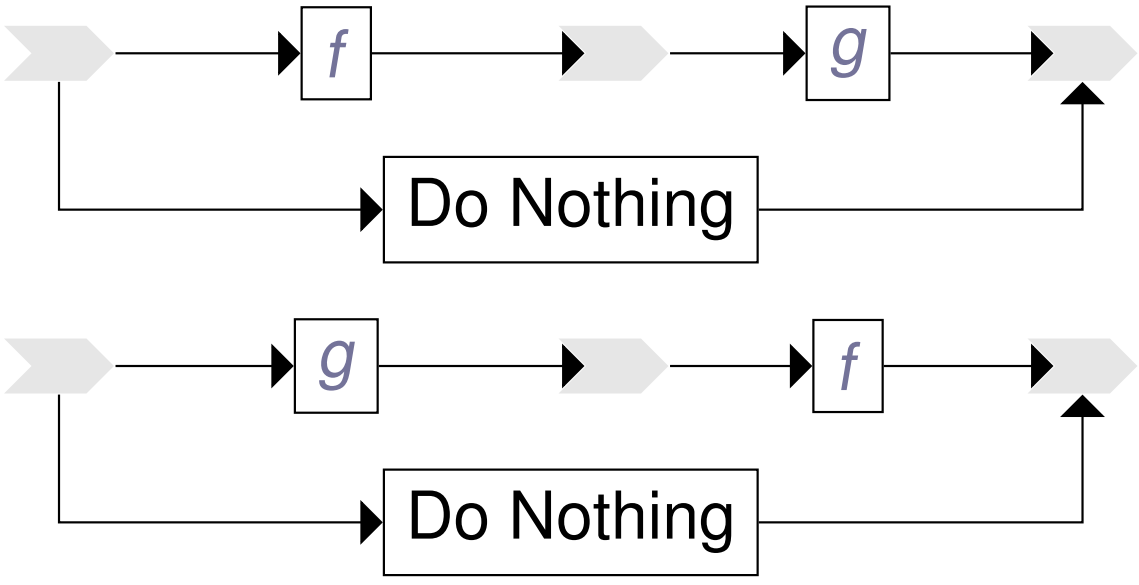
Busy Doing Nothing



Busy Doing Nothing



Busy Doing Nothing



f invertible

$\Longleftrightarrow$

$\exists g$: gf and fg are both "Do Nothing"

In Matrix–Speak

The “Do Nothing” process:

$$y_1 = x_1, y_2 = x_2, \ldots, y_n = x_n$$

In Matrix–Speak

The “Do Nothing” process:

$$y_1 = x_1, y_2 = x_2, \ldots, y_n = x_n$$

Representing matrix: (Technically, one for each size n)

$$I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

In Matrix–Speak

The “Do Nothing” process:

$$y_1 = x_1, y_2 = x_2, \ldots, y_n = x_n$$

Representing matrix: (Technically, one for each size n)

$$I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Definition or Theorem

A matrix A is
invertible
if there is a matrix B such that
$AB = I_m, \qquad BA = I_n$

Why Bother?

Basic Problem

Solve

$$A\mathbf{x} = \mathbf{b}$$

Why Bother?

Basic Problem

Solve

$$A\mathbf{x} = \mathbf{b}$$

Output of process A is $\mathbf{b}$, what was the input?

Why Bother?

Basic Problem

Solve

$$A\mathbf{x} = \mathbf{b}$$

Output of process A is $\mathbf{b}$, what was the input?

Obvious Use of Invertibility

$$A\mathbf{x} = \quad \mathbf{b}$$

1. If A is invertible, inverse B . . .

Why Bother?

Basic Problem

Solve

$$A\mathbf{x} = \mathbf{b}$$

Output of process A is $\mathbf{b}$, what was the input?

Obvious Use of Invertibility

$$BA\mathbf{x} = B\mathbf{b}$$

1. If A is invertible, inverse B . . . apply B

Why Bother?

Basic Problem

Solve

$$A\mathbf{x} = \mathbf{b}$$

Output of process A is $\mathbf{b}$, what was the input?

Obvious Use of Invertibility

$$I_n\mathbf{x} = B\mathbf{b}$$

1. If A is invertible, inverse B . . . apply B
2. Use $BA = I_n$

Why Bother?

Basic Problem

Solve

$$A\mathbf{x} = \mathbf{b}$$

Output of process A is $\mathbf{b}$, what was the input?

Obvious Use of Invertibility

$$\mathbf{x} = B\mathbf{b}$$

1. If A is invertible, inverse B . . . apply B
2. Use $BA = I_n$
3. I_n means “do nothing”

Bother?

Ba blem

Solve

$$A\mathbf{x} = \mathbf{b}$$

Output of process what input?

Obvious Use of Inve

$$\mathbf{x} = B\mathbf{b}$$

1. If ertible, inverse B . . . apply B

$BA = I_n$

I_n means “do nothing”

The True Meaning of Invertible

Exercise

10130927 10141007

The True Meaning of Invertible

Exercise

10130927 10141007

Tasks:

1. Factor 1st number

The True Meaning of Invertible

Exercise

10130927 10141007

Tasks:

1. Factor 1st number
2. Factor 2nd number

The True Meaning of Invertible

Exercise

10130927 10141007

Tasks:

1. Factor 1st number
2. Factor 2nd number
3. Multiply the numbers

The True Meaning of Invertible

Exercise

10130927 10141007

Tasks:

1. Factor 1st number
2. Factor 2nd number
3. Multiply the numbers

Point

It is easier to do than undo

True Use of Invertibility

A invertible

$A\mathbf{b} = \mathbf{c}$

$\Longrightarrow$

$\mathbf{b}$ and $\mathbf{c}$ represent same data

True Use of Invertibility

A invertible

$A\mathbf{b} = \mathbf{c}$

$\Longrightarrow$

$\mathbf{b}$ and $\mathbf{c}$ represent same data

Another View

$\mathbf{b}$ and $\mathbf{c}$ are

two ways

of seeing the same thing

True Use of Invertibility

A invertible

$A\mathbf{b} = \mathbf{c}$

$\Longrightarrow$

$\mathbf{b}$ and $\mathbf{c}$ represent same data

Another View

$\mathbf{b}$ and $\mathbf{c}$ are
two ways
of seeing the same thing

Consequence

Use invertible matrices to change your point of view to something nicer

True Use of Invertibility

$$A\mathbf{x} = \mathbf{b}$$

Do Not

Try to find inverse of A — factoring

Do

Use invertible matrices so A looks nice — multiplying

True Use of Invertibility

$$A\mathbf{x} = \mathbf{b}$$

Do Not

Try to find inverse of A — factoring

Do

Use invertible matrices so A looks nice — multiplying

Remember

Do the same to both sides!

Finding Invertible Matrices

Invertible

$$A \text{ invertible}$$
$$\Longleftrightarrow$$
$$\exists B : AB = I_m \text{ and } BA = I_n$$

Finding Invertible Matrices

Invertible

$$A \text{ invertible}$$
$$\Longleftrightarrow$$
$$\exists B : AB = I_m \text{ and } BA = I_n$$

Difficulty

Proving invertible means finding B!

Obvious Examples

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Obvious Examples

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad I_n I_n = I_n$$

("do nothing" and "do nothing again" = "do nothing")

Obvious Examples

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad I_n I_n = I_n$$

("do nothing" and "do nothing again" = "do nothing")

$$\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Obvious Examples

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad I_n I_n = I_n$$

("do nothing" and "do nothing again" = "do nothing")

$$\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$(x_1, x_2, \ldots, x_n) \mapsto (x_2, x_1, \ldots, x_n)$$

reversed by

$$(y_1, y_2, \ldots, y_n) \mapsto (y_2, y_1, \ldots, y_n)$$

Examples

$$\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Examples

$$\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 1/2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Examples

$$\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 1/2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$(x_1, x_2, \ldots, x_n) \mapsto (2x_1, x_2, \ldots, x_n)$$

reversed by

$$(y_1, y_2, \ldots, y_n) \mapsto (1/2 y_1, y_2, \ldots, y_n)$$

Examples

$$\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 1/2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$(x_1, x_2, \ldots, x_n) \mapsto (2x_1, x_2, \ldots, x_n)$$

reversed by

$$(y_1, y_2, \ldots, y_n) \mapsto (1/2 y_1, y_2, \ldots, y_n)$$

More generally, for $\lambda_i \neq 0$

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Examples

$$\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 1/2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$(x_1, x_2, \ldots, x_n) \mapsto (2x_1, x_2, \ldots, x_n)$$
reversed by
$$(y_1, y_2, \ldots, y_n) \mapsto (1/2 y_1, y_2, \ldots, y_n)$$

More generally, for $\lambda_i \neq 0$

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\lambda_n \end{bmatrix}$$

More Examples

$$\begin{bmatrix} 1 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

More Examples

$$\begin{bmatrix} 1 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 1 & -1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

More Examples

$$\begin{bmatrix} 1 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ has inverse } \begin{bmatrix} 1 & -1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1 + x_2, x_2, \ldots, x_n)$$

reversed by

$$(y_1, y_2, \ldots, y_n) \mapsto (y_1 - y_2, y_2, \ldots, y_n)$$

Row Operations are Invertible Matrices

Key Fact

Row op: $A \to B$

$\Longrightarrow$

$B = EA$

with

E invertible

Two Key Facts

1. There is only one way to invert an invertible process

Two Key Facts

1. There is only one way to invert an invertible process
   $\Longrightarrow$ if A is invertible, its inverse is unique

Two Key Facts

1. There is only one way to invert an invertible process
   $\Longrightarrow$ if A is invertible, its inverse is unique
   Notation: A^{-1}

Two Key Facts

1. There is only one way to invert an invertible process
   $\Longrightarrow$ if A is invertible, its inverse is unique
   Notation: A^{-1}
2. The composition of two invertible processes is invertible

Two Key Facts

1. There is only one way to invert an invertible process
   $\Longrightarrow$ if A is invertible, its inverse is unique
   Notation: A^{-1}
2. The composition of two invertible processes is invertible
   $\Longrightarrow$ if A, B invertible and composable, then AB is invertible

Two Key Facts

1. There is only one way to invert an invertible process
   $\Longrightarrow$ if A is invertible, its inverse is unique
   Notation: A^{-1}
2. The composition of two invertible processes is invertible
   $\Longrightarrow$ if A, B invertible and composable, then AB is invertible
   $(AB)^{-1} = B^{-1}A^{-1}$

Two Key Facts

1. There is only one way to invert an invertible process
   $\Longrightarrow$ if A is invertible, its inverse is unique
   Notation: A^{-1}
2. The composition of two invertible processes is invertible
   $\Longrightarrow$ if A, B invertible and composable, then AB is invertible
   $(AB)^{-1} = B^{-1}A^{-1}$
   ("take off shoes, then socks")

Important Implications

Row Reduction

If $A \to B$ via row reduction then

A invertible $\iff$ B invertible

Important Implications

Row Reduction

If $A \to B$ via row reduction then
A invertible $\iff$ B invertible

If $A \to I_m$ via row reduction then A is invertible

Important Implications

Row Reduction

If $A \to B$ via row reduction then
A invertible $\iff$ B invertible

If $A \to I_m$ via row reduction then A is invertible

Can we say more?

The Result of Row Reduction

Full Gauss–Jordan Row Reduction

Example outcomes:

The Result of Row Reduction

Full Gauss–Jordan Row Reduction

Example outcomes:

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix}$$

The Result of Row Reduction

Full Gauss–Jordan Row Reduction

Example outcomes:

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix}$$

Not invertible:

$$A \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Result of Row Reduction

Full Gauss–Jordan Row Reduction

Example outcomes:

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix}$$

Not invertible:

$$A \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Inevitable if $\text{rows}(A) < \text{cols}(A)$

The Result of Row Reduction

Full Gauss–Jordan Row Reduction

Example outcomes:

The Result of Row Reduction

Full Gauss–Jordan Row Reduction

Example outcomes:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The Result of Row Reduction

Full Gauss–Jordan Row Reduction

Example outcomes:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Not invertible:

$$Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

has no solution.

The Result of Row Reduction

Full Gauss–Jordan Row Reduction

Example outcomes:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Not invertible:

$$Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

has no solution.

Inevitable if $\text{rows}(A) > \text{cols}(A)$

Important Implications

Invertible Size

$$A \text{ invertible} \Longrightarrow A \text{ square}$$

Important Implications

Invertible Size

$$A \text{ invertible} \Longrightarrow A \text{ square}$$

Other direction false:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is not invertible.

Square Row Reduction Results

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Square Row Reduction Results

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The first is the only matrix such that:

1. A is invertible

Square Row Reduction Results

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The first is the only matrix such that:

1. A is invertible
2. $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$

Square Row Reduction Results

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The first is the only matrix such that:

1. A is invertible
2. $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$
3. $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$

Square Row Reduction Results

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The first is the only matrix such that:

1. A is invertible
2. $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$
3. $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{0}$ has a unique solution

Square Row Reduction Results

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The first is the only matrix such that:

1. A is invertible
2. $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$
3. $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{0}$ has a unique solution

This is independent of size of square matrix

Other Methods

When is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

invertible?

Other Methods

When is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

invertible?
Can solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$

Other Methods

When is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

invertible?
Can solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$
Apply GE to

$$\left[\begin{array}{cc|c} a & b & b_1 \\ c & d & b_2 \end{array}\right]$$

Other Methods

When is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

invertible?
Can solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$
Apply GE to

$$\left[\begin{array}{cc|c} a & b & b_1 \\ c & d & b_2 \end{array}\right] \xrightarrow[a \neq 0]{R_2 - {}^{c}/_{a} R_1} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & d - {}^{cb}/_{a} & b_2 - {}^{c}/_{a} b_1 \end{array}\right]$$

Other Methods

When is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

invertible?
Can solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$
Apply GE to

$$\left[\begin{array}{cc|c} a & b & b_1 \\ c & d & b_2 \end{array}\right] \xrightarrow[a \neq 0]{R_2 - {}^{c}/_{a} R_1} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & d - {}^{cb}/_{a} & b_2 - {}^{c}/_{a} b_1 \end{array}\right]$$

$$\xrightarrow{R_2 \to aR_2} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & ad - cb & ab_2 - cb_1 \end{array}\right]$$

Other Methods

When is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

invertible?
Can solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$
Apply GE to

$$\left[\begin{array}{cc|c} a & b & b_1 \\ c & d & b_2 \end{array}\right] \xrightarrow[a \neq 0]{R_2 - {}^{c}/_{a} R_1} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & d - {}^{cb}/_{a} & b_2 - {}^{c}/_{a} b_1 \end{array}\right]$$

$$\xrightarrow{R_2 \to aR_2} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & ad - cb & ab_2 - cb_1 \end{array}\right]$$

If $a \neq 0$ then A invertible $\iff$ $ad - bc \neq 0$

Other Methods

When is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

invertible?
Can solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$
Apply GE to

$$\left[\begin{array}{cc|c} a & b & b_1 \\ c & d & b_2 \end{array}\right] \xrightarrow[a \neq 0]{R_2 - {}^{c}/_{a} R_1} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & d - {}^{cb}/_{a} & b_2 - {}^{c}/_{a} b_1 \end{array}\right]$$

$$\xrightarrow{R_2 \to aR_2} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & ad - cb & ab_2 - cb_1 \end{array}\right]$$

If $a \neq 0$ then A invertible $\iff$ $ad - bc \neq 0$
If $c \neq 0$ then swap so A invertible $\iff$ $cb - ad \neq 0$

Other Methods

When is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

invertible?
Can solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$
Apply GE to

$$\left[\begin{array}{cc|c} a & b & b_1 \\ c & d & b_2 \end{array}\right] \xrightarrow[a \neq 0]{R_2 - c/a R_1} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & d - cb/a & b_2 - c/a b_1 \end{array}\right]$$

$$\xrightarrow{R_2 \to aR_2} \left[\begin{array}{cc|c} a & b & b_1 \\ 0 & ad - cb & ab_2 - cb_1 \end{array}\right]$$

If $a \neq 0$ then A invertible $\iff$ $ad - bc \neq 0$
If $c \neq 0$ then swap so A invertible $\iff$ $cb - ad \neq 0$
If $a = c = 0$ then A not invertible and $cb - ad = 0$

It's that Strange Sense of Deja Vu

y_1, y_2 solutions of an ODE
linearly independent

$$\Longleftrightarrow$$

$$y_1 y_2' - y_1' y_2 \neq 0$$

$$A$$

invertible

$$\Longleftrightarrow \quad ad - bc \neq 0$$

It's that Strange Sense of Deja Vu

y_1, y_2 solutions of an ODE
linearly independent

$$\Longleftrightarrow$$

$$y_1 y_2' - y_1' y_2 \neq 0$$

$$A$$

invertible

$$\Longleftrightarrow \quad ad - bc \neq 0$$

Not a coincidence!

Determining Invertibility

Definition

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant of A is

$$\det(A) := \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

Determining Invertibility

Definition

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant of A *is*

$$\det(A) := \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

Lemma

$$A \text{ invertible} \iff \det(A) \neq 0$$

Bigger and . . . Better?

Sort of Definition

Square matrices of any size have determinants.

Bigger and . . . Better?

Sort of Definition

Square matrices of any size have determinants.

Exact definition not exactly . . . memorable:

$$\det(A) = \sum_{\sigma \in \Sigma_n} (-1)^{\sigma} a_{11\sigma} \cdots a_{nn\sigma}$$

Bigger and . . . Better?

Sort of Definition

Square matrices of any size have determinants.

Exact definition not exactly . . . memorable:

$$\det(A) = \sum_{\sigma \in \Sigma_n} (-1)^{\sigma} a_{11\sigma} \cdots a_{nn\sigma}$$

Best: Learn rules for computation.

Don't Know? Ask Gauss

Key Properties for Computation

1. $\det(I_n) = 1$

Don't Know? Ask Gauss

Key Properties for Computation

1. $\det(I_n) = 1$
2. $\det(AB) = \det(A)\det(B)$

Don't Know? Ask Gauss

Key Properties for Computation

1. $\det(I_n) = 1$
2. $\det(AB) = \det(A)\det(B)$
3. Triangular matrices:

$$\begin{vmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{vmatrix} = 2 \times 5 \times 7 = 70$$

Don't Know? Ask Gauss

Key Properties for Computation

1. $\det(I_n) = 1$
2. $\det(AB) = \det(A)\det(B)$
3. Triangular matrices:

$$\begin{vmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{vmatrix} = 2 \times 5 \times 7 = 70$$

4. Row Swaps:

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = - \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix}$$

Don't Know? Ask Gauss

Key Properties for Computation

1. $\det(I_n) = 1$
2. $\det(AB) = \det(A)\det(B)$
3. Triangular matrices:

$$\begin{vmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{vmatrix} = 2 \times 5 \times 7 = 70$$

4. Row Swaps:

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = - \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix}$$

5. Row Multipliers:

$$\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$$

Note: only one row affected.

Don't Know? Ask Gauss

6. Row Adds:

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}$$

No change!

Don't Know? Ask Gauss

6. Row Adds:

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}$$

   No change!

7. Works for columns too.

Don't Know? Ask Gauss

6. Row Adds:

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}$$

   No change!

7. Works for columns too.

Conclusion

Whenever you need to compute something, use GE!

Don't Know? Ask Gauss

6. Row Adds:

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}$$

   No change!

7. Works for columns too.

Conclusion

Whenever you need to compute something, use GE!

Mnemonic

$$\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} = -2$$

Example

$$\begin{vmatrix} 0 & 2 & 3 \\ 3 & 4 & 0 \\ 1 & 6 & 2 \end{vmatrix}$$

Example

$$\begin{vmatrix} 0 & 2 & 3 \\ 3 & 4 & 0 \\ 1 & 6 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 6 & 2 \\ 3 & 4 & 0 \\ 0 & 2 & 3 \end{vmatrix} (R_1 \leftrightarrow R_3)$$

Example

$$\begin{vmatrix} 0 & 2 & 3 \\ 3 & 4 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ 3 & 4 & 0 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_1 \leftrightarrow R_3)$$

$$= -\begin{vmatrix} 1 & 6 & 2 \\ 0 & -14 & -6 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_2 - 3R_1)$$

Example

$$\begin{vmatrix} 0 & 2 & 3 \\ 3 & 4 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ 3 & 4 & 0 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_1 \leftrightarrow R_3)$$

$$= -\begin{vmatrix} 1 & 6 & 2 \\ 0 & -14 & -6 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_2 - 3R_1) \qquad = -2\begin{vmatrix} 1 & 3 & 2 \\ 0 & -7 & -6 \\ 0 & 1 & 3 \end{vmatrix} \quad (C_2 \to {}^1\!/_2 C_2)$$

Example

$$\begin{vmatrix} 0 & 2 & 3 \\ 3 & 4 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ 3 & 4 & 0 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_1 \leftrightarrow R_3)$$

$$= -\begin{vmatrix} 1 & 6 & 2 \\ 0 & -14 & -6 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_2 - 3R_1) \qquad = -2\begin{vmatrix} 1 & 3 & 2 \\ 0 & -7 & -6 \\ 0 & 1 & 3 \end{vmatrix} \quad (C_2 \to {}^1\!/_2 C_2)$$

$$= 2\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & -7 & -6 \end{vmatrix} \quad (R_2 \leftrightarrow R_3)$$

Example

$$\begin{vmatrix} 0 & 2 & 3 \\ 3 & 4 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ 3 & 4 & 0 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_1 \leftrightarrow R_3)$$

$$= -\begin{vmatrix} 1 & 6 & 2 \\ 0 & -14 & -6 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_2 - 3R_1) \qquad = -2\begin{vmatrix} 1 & 3 & 2 \\ 0 & -7 & -6 \\ 0 & 1 & 3 \end{vmatrix} \quad (C_2 \to {}^1\!/_2 C_2)$$

$$= 2\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & -7 & -6 \end{vmatrix} \quad (R_2 \leftrightarrow R_3) \qquad = 2\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 15 \end{vmatrix} \quad (R_3 + 7R_2)$$

Example

$$\begin{vmatrix} 0 & 2 & 3 \\ 3 & 4 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ 3 & 4 & 0 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_1 \leftrightarrow R_3)$$

$$= -\begin{vmatrix} 1 & 6 & 2 \\ 0 & -14 & -6 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_2 - 3R_1) \qquad = -2\begin{vmatrix} 1 & 3 & 2 \\ 0 & -7 & -6 \\ 0 & 1 & 3 \end{vmatrix} \quad (C_2 \to {}^1\!/_2 C_2)$$

$$= 2\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & -7 & -6 \end{vmatrix} \quad (R_2 \leftrightarrow R_3) \qquad = 2\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 15 \end{vmatrix} \quad (R_3 + 7R_2)$$

$$= 2 \times 15 = 30$$

Summary

- Use invertible matrices to simplify problems
- Invertible means “equivalent point of view”
- Lots of tests, most useful:
  A invertible $\iff$ A row reduces to I_n
- Computing determinants: use row and column rules to simplify