TMA4115 Matematikk 3

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Lecture 12: Mathematics’ Marvellous Matrices

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Key Points

- Process $\longleftrightarrow$ Matrix:

  Obvious manipulations of processes

  $$\longleftrightarrow$$

  Matrix manipulations

- The true meaning of “invertible”

Detailed Recap

Definition

A
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(of numbers)

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$$\begin{bmatrix} 1 & 3 & -4 & 5 \\ \pi & e^{\pi} & -\pi^{e} & -e \\ 0 & 28 & -2 & 0 \end{bmatrix} \quad \checkmark$$

Detailed Recap

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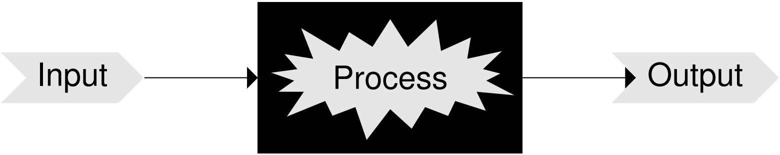
Meaningful Matrices

Key Point

Matrices encode certain processes



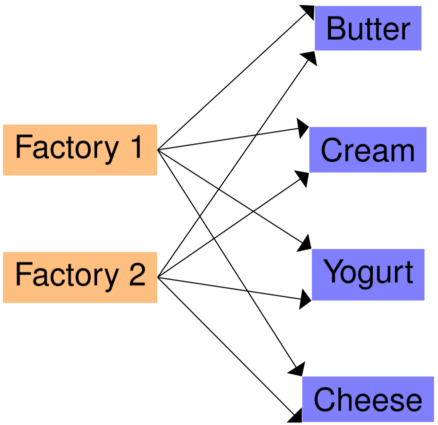
$$x_1, \ldots, x_n \longrightarrow \begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \quad \ddots \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{matrix}$$

Dairy Production

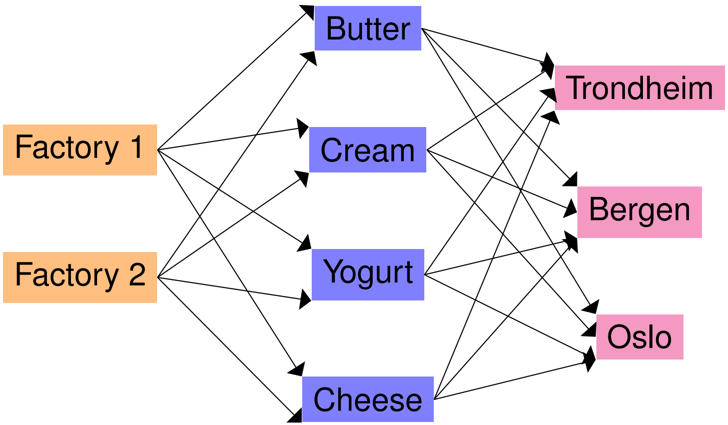
Factory 1

Factory 2

Dairy Production



Dairy Production



The Processes

Factories to Products

Products: how much of factory 2's output is butter?

	factory 1	factory 2
butter	.3	.1
cream	.1	.0
yogurt	.2	.4
cheese	.4	.5

Products to Cities

How much of each tonne of butter goes to Trondheim?

	butter	cream	yogurt	cheese
Trondheim	.22	.25	.27	.21
Bergen	.30	.30	.30	.30
Oslo	.48	.45	.43	.49

From Factory to City

How much “stuff” goes from Factory 1 to Trondheim?

$$\begin{bmatrix} & \text{F 1} & \text{F 2} \\ \text{T} & & \\ \text{B} & & \\ \text{O} & & \end{bmatrix} = \begin{bmatrix} .22 & .25 & .27 & .21 \\ .30 & .30 & .30 & .30 \\ .48 & .45 & .43 & .49 \end{bmatrix} \begin{bmatrix} .3 & .1 \\ .1 & .0 \\ .2 & .4 \\ .4 & .5 \end{bmatrix}$$

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$$.22 \times .3 + .25 \times .1 + .27 \times .2 + .21 \times .4$$

From Factory to City

How much “stuff” goes from Factory 1 to Trondheim?

$$\begin{bmatrix} & \text{F 1} & \text{F 2} \\ \text{T} & .229 & \\ \text{B} & & \\ \text{O} & & \end{bmatrix} = \begin{bmatrix} .22 & .25 & .27 & .21 \\ .30 & .30 & .30 & .30 \\ .48 & .45 & .43 & .49 \end{bmatrix} \begin{bmatrix} .3 & .1 \\ .1 & .0 \\ .2 & .4 \\ .4 & .5 \end{bmatrix}$$

$$.22 \times .3 + .25 \times .1 + .27 \times .2 + .21 \times .4 = .229$$

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How much "stuff" goes from Factory 1 to Trondheim?

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$$.22 \times .1 + .25 \times .0 + .27 \times .4 + .21 \times .5$$

From Factory to City

How much “stuff” goes from Factory 1 to Trondheim?

$$\begin{bmatrix} & \text{F 1} & \text{F 2} \\ \text{T} & .229 & .235 \\ \text{B} & & \\ \text{O} & & \end{bmatrix} = \begin{bmatrix} .22 & .25 & .27 & .21 \\ .30 & .30 & .30 & .30 \\ .48 & .45 & .43 & .49 \end{bmatrix} \begin{bmatrix} .3 & .1 \\ .1 & .0 \\ .2 & .4 \\ .4 & .5 \end{bmatrix}$$

$$.22 \times .1 + .25 \times .0 + .27 \times .4 + .21 \times .5 = .235$$

From Factory to City

How much “stuff” goes from Factory 1 to Trondheim?

$$\begin{bmatrix} & \text{F 1} & \text{F 2} \\ \text{T} & .229 & .235 \\ \text{B} & & \\ \text{O} & & \end{bmatrix} = \begin{bmatrix} .22 & .25 & .27 & .21 \\ .30 & .30 & .30 & .30 \\ .48 & .45 & .43 & .49 \end{bmatrix} \begin{bmatrix} .3 & .1 \\ .1 & .0 \\ .2 & .4 \\ .4 & .5 \end{bmatrix}$$

$$.30 \times .3 + .30 \times .1 + .30 \times .2 + .30 \times .4$$

From Factory to City

How much "stuff" goes from Factory 1 to Trondheim?

$$\begin{bmatrix} & \text{F 1} & \text{F 2} \\ \text{T} & .229 & .235 \\ \text{B} & .300 & \\ \text{O} & & \end{bmatrix} = \begin{bmatrix} .22 & .25 & .27 & .21 \\ .30 & .30 & .30 & .30 \\ .48 & .45 & .43 & .49 \end{bmatrix} \begin{bmatrix} .3 & .1 \\ .1 & .0 \\ .2 & .4 \\ .4 & .5 \end{bmatrix}$$

$$.30 \times .3 + .30 \times .1 + .30 \times .2 + .30 \times .4 = .300$$

From Factory to City

How much “stuff” goes from Factory 1 to Trondheim?

$$\begin{bmatrix} & \text{F 1} & \text{F 2} \\ \text{T} & .229 & .235 \\ \text{B} & .300 & \\ \text{O} & & \end{bmatrix} = \begin{bmatrix} .22 & .25 & .27 & .21 \\ .30 & .30 & .30 & .30 \\ .48 & .45 & .43 & .49 \end{bmatrix} \begin{bmatrix} .3 & .1 \\ .1 & .0 \\ .2 & .4 \\ .4 & .5 \end{bmatrix}$$

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$$.48 \times .3 + .45 \times .1 + .43 \times .2 + .49 \times .4$$

From Factory to City

How much "stuff" goes from Factory 1 to Trondheim?

$$\begin{bmatrix} & \text{F 1} & \text{F 2} \\ \text{T} & .229 & .235 \\ \text{B} & .300 & .300 \\ \text{O} & .471 & \end{bmatrix} = \begin{bmatrix} .22 & .25 & .27 & .21 \\ .30 & .30 & .30 & .30 \\ .48 & .45 & .43 & .49 \end{bmatrix} \begin{bmatrix} .3 & .1 \\ .1 & .0 \\ .2 & .4 \\ .4 & .5 \end{bmatrix}$$

$$.48 \times .3 + .45 \times .1 + .43 \times .2 + .49 \times .4 = .465$$

From Factory to City

How much “stuff” goes from Factory 1 to Trondheim?

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From Factory to City

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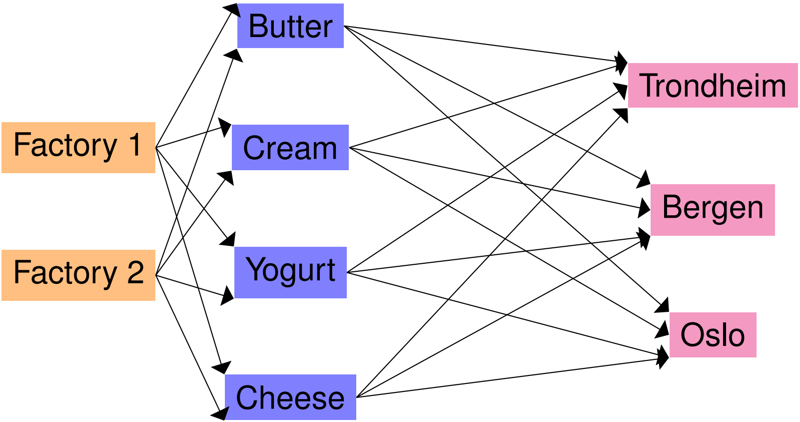
Add in Depots



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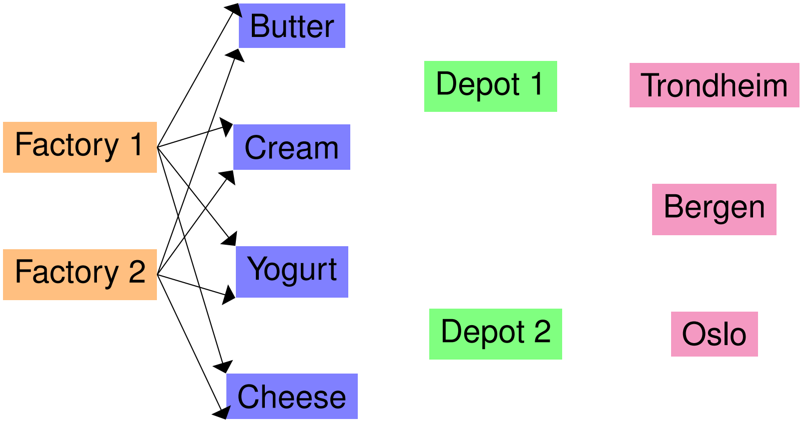

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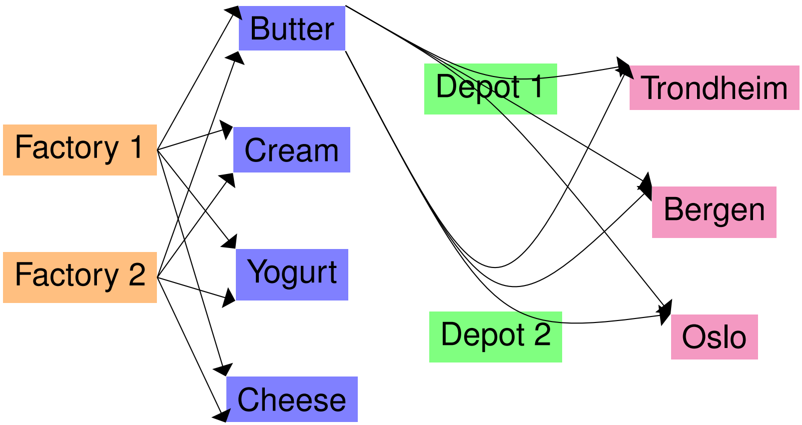

Partitioning Processes

Factory to City

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Factory to City

via Depot 1

$$\begin{bmatrix} .087 & .105 \\ .087 & .105 \\ .116 & .140 \end{bmatrix}$$

via Depot 2

$$\begin{bmatrix} .142 & .030 \\ .213 & .195 \\ .355 & .325 \end{bmatrix}$$

Partitioning Processes

Factory to City

via Depot 1

$$\begin{bmatrix} .087 & .105 \\ .087 & .105 \\ .116 & .140 \end{bmatrix}$$

via Depot 2

$$\begin{bmatrix} .142 & .030 \\ .213 & .195 \\ .355 & .325 \end{bmatrix}$$

Total

Factory 1 to Trondheim
=
Factory 1 to Trondheim via Depot 1
+
Factory 1 to Trondheim via Depot 2

$$\begin{bmatrix} .087 & .105 \\ .087 & .105 \\ .116 & .140 \end{bmatrix} + \begin{bmatrix} .142 & .030 \\ .213 & .195 \\ .355 & .325 \end{bmatrix} = \begin{bmatrix} .229 & .235 \\ .300 & .300 \\ .471 & .465 \end{bmatrix}$$

Combining Processes

1. One process follows the other: multiply matrices
2. Two processes partition another: add matrices

A Secret Process

Question

How do we increase factory output?

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How do we increase factory output?

Answer

Increase Milk Production!

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$$\text{Milk} \longrightarrow \text{Produce}$$

A Secret Process

Question

How do we increase factory output?

Answer

Increase Milk Production!

- Factory output proportional to milk production
- View factory output, (p_1, p_2), as a process:

$$\text{Milk} \longrightarrow \text{Produce}$$

- Representing matrix is:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

Vectors

Definition

A column vector
is a matrix with one column

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Remark

1. Column vectors written as $\vec{u}$ or $\mathbf{u}$ or $\underline{u}$
2. View as "initial process": defining the input data for a process
3. Matrix $\times$ column vector = column vector
   because output of one process is input of next

Matrices and Column Vectors

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 1 & 3 & -4 & 5 \\ 2 & 8 & 0 & 0 \\ 2 & 1 & -3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

Matrices and Column Vectors

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$$\begin{bmatrix} 1 \\ -10 \\ \end{bmatrix} = \begin{bmatrix} 1 & 3 & -4 & 5 \\ 2 & 8 & 0 & 0 \\ 2 & 1 & -3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

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Matrices and Column Vectors

$$\begin{bmatrix} 1 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -4 & 5 \\ 2 & 8 & 0 & 0 \\ 2 & 1 & -3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

Column Vectors and Linear Systems

Linear System

Find x, y, z such that $$\begin{aligned} x + 2y + z &= 2 \\ 2x - 3y + z &= 3 \\ -x + y + 2z &= 4 \end{aligned}$$

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Column Vectors and Linear Systems

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Find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Same question!

A Convenient Notation

Write generic linear system as:

$$A\vec{x} = \vec{b}$$

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Composition order:

- A and B represent processes

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Write generic linear system as:

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Composition order:

- A and B represent processes
- feed outputs of A into B to get new process, matrix C
- write $C = BA$
- substitute into notation for linear system:

$$C\vec{x} = BA\vec{x}$$

do A first and then B

Factory to Depot

Passing Through the Depots

$$\begin{bmatrix} & F1 & F2 \\ D1 & .29 & .35 \\ D2 & .71 & .65 \end{bmatrix}$$

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Recall: matrix records proportions

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Recall: matrix records proportions

Question

Can we find out the productivity of the factories by measuring what arrives at the depots?

Factory to Depot

Passing Through the Depots

$$\begin{bmatrix} & F1 & F2 \\ D1 & .29 & .35 \\ D2 & .71 & .65 \end{bmatrix}$$

Recall: matrix records proportions

Question

Can we find out the productivity of the factories by measuring what arrives at the depots?

Solve:

$$\begin{aligned} .29p_1 + .35p_2 &= d_1 \\ .71p_1 + .65p_2 &= d_2 \end{aligned}$$

Solution

$$\left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .71 & .65 & d_2 \end{array}\right]$$

Solution

$$\left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .71 & .65 & d_2 \end{array}\right] \xrightarrow{R_2 \to .29 R_2} \left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .2059 & .1885 & .29 d_2 \end{array}\right]$$

Solution

$$\left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .71 & .65 & d_2 \end{array}\right] \xrightarrow{R_2 \to .29 R_2} \left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .2059 & .1885 & .29 d_2 \end{array}\right]$$

$$\xrightarrow{R_2 - .71 R_1} \left[\begin{array}{cc|c} .29 & .35 & d_1 \\ 0 & -.06 & .29 d_2 - .71 d_1 \end{array}\right]$$

Solution

$$\left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .71 & .65 & d_2 \end{array}\right] \xrightarrow{R_2 \to .29 R_2} \left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .2059 & .1885 & .29 d_2 \end{array}\right]$$

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$$\xrightarrow{R_1 \to -.06 R_1} \left[\begin{array}{cc|c} -.0174 & -.0210 & -.06 d_1 \\ 0 & -.06 & .29 d_2 - .71 d_1 \end{array}\right]$$

Solution

$$\left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .71 & .65 & d_2 \end{array}\right] \xrightarrow{R_2 \to .29 R_2} \left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .2059 & .1885 & .29 d_2 \end{array}\right]$$

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$$\xrightarrow{R_1 \to -.06 R_1} \left[\begin{array}{cc|c} -.0174 & -.0210 & -.06 d_1 \\ 0 & -.06 & .29 d_2 - .71 d_1 \end{array}\right]$$

$$\xrightarrow{R_1 - .35 R_2} \left[\begin{array}{cc|c} -.0174 & 0 & -.1885 d_1 - .1015 d_2 \\ 0 & -.06 & .29 d_2 - .71 d_1 \end{array}\right]$$

Solution

$$\left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .71 & .65 & d_2 \end{array}\right] \xrightarrow{R_2 \to .29 R_2} \left[\begin{array}{cc|c} .29 & .35 & d_1 \\ .2059 & .1885 & .29 d_2 \end{array}\right]$$

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$$\xrightarrow{R_1 \to -.06 R_1} \left[\begin{array}{cc|c} -.0174 & -.0210 & -.06 d_1 \\ 0 & -.06 & .29 d_2 - .71 d_1 \end{array}\right]$$

$$\xrightarrow{R_1 - .35 R_2} \left[\begin{array}{cc|c} -.0174 & 0 & -.1885 d_1 - .1015 d_2 \\ 0 & -.06 & .29 d_2 - .71 d_1 \end{array}\right]$$

$$\xrightarrow{R_1 \to .29^{-1} R_1} \left[\begin{array}{cc|c} -.06 & 0 & .65 d_1 - .35 d_2 \\ 0 & -.06 & .29 d_2 - .71 d_1 \end{array}\right]$$

Back and Forth

$$.29p_1 + .35p_2 = d_1 \qquad \frac{-100}{6}\Big(.65d_1 - .35d_2\Big) = p_1$$

$$.71p_1 + .65p_2 = d_2 \qquad \frac{-100}{6}\Big(-.71d_1 + .29d_2\Big) = p_2$$

Back and Forth

$$\begin{aligned} .29p_1 + .35p_2 &= d_1 \\ .71p_1 + .65p_2 &= d_2 \end{aligned} \qquad \begin{aligned} \tfrac{-100}{6}\Big(.65d_1 - .35d_2\Big) &= p_1 \\ \tfrac{-100}{6}\Big(-.71d_1 + .29d_2\Big) &= p_2 \end{aligned}$$

Remark

1. Can figure out what leaves factories from what arrives at depots.

Back and Forth

$$
\begin{aligned}
.29p_1 + .35p_2 &= d_1 \\
.71p_1 + .65p_2 &= d_2
\end{aligned}
\qquad
\begin{aligned}
\tfrac{-100}{6}\Big(.65d_1 - .35d_2\Big) &= p_1 \\
\tfrac{-100}{6}\Big(-.71d_1 + .29d_2\Big) &= p_2
\end{aligned}
$$

Remark

1. Can figure out what leaves factories from what arrives at depots.
2. Can arrange for any desired arrival amounts by adjusting productions.

Back and Forth

$$\begin{aligned} .29p_1 + .35p_2 &= d_1 \\ .71p_1 + .65p_2 &= d_2 \end{aligned} \qquad \begin{aligned} \tfrac{-100}{6}\Big(.65d_1 - .35d_2\Big) &= p_1 \\ \tfrac{-100}{6}\Big(-.71d_1 + .29d_2\Big) &= p_2 \end{aligned}$$

Remark

1. Can figure out what leaves factories from what arrives at depots.
2. Can arrange for any desired arrival amounts by adjusting productions.
3. No presumption of cause or effect.

Invertible

Definition

A process is said to be
invertible
if

1. *Each input is uniquely determined by its output*
2. *Each potential output is possible*

A matrix is
invertible
if
it represents an invertible process

Invertible Examples

1. $$\begin{bmatrix} .29 & .35 \\ .71 & .65 \end{bmatrix}$$

   represents an invertible process

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1. $$\begin{bmatrix} .29 & .35 \\ .71 & .65 \end{bmatrix}$$

   represents an invertible process

2. $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

   is not invertible: $(0, 0, 0)$ and $(1, -2, 1)$ both go to $(0, 0)$

Invertible Examples

1. $$\begin{bmatrix} .29 & .35 \\ .71 & .65 \end{bmatrix}$$

   represents an invertible process

2. $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

   is not invertible: $(0, 0, 0)$ and $(1, -2, 1)$ both go to $(0, 0)$

3. $$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   is not invertible: no way to get $(1, 0)$

An Incredibly Important Process

Introducing

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Representing matrix:

$$I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

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(Technically, one for each size n)

Inverses

Rough Definition

An inverse of a process is another process such that composing either way around results in the “do nothing” process.

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Lemma

A matrix is invertible if and only if it has an inverse.

Computing Inverses

Does

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

have an inverse?

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Let's see if it does.

Simultaneous Gaussian Elimination

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 4 & 6 & 0 & 0 & 1 \end{bmatrix}$$

Simultaneous Gaussian Elimination

$$\begin{bmatrix}1 & 2 & 3 & 1 & 0 & 0\\2 & 3 & 4 & 0 & 1 & 0\\3 & 4 & 6 & 0 & 0 & 1\end{bmatrix} \xrightarrow[R_3-{\color{red}3}R_1]{R_2-{\color{red}2}R_1} \begin{bmatrix}1 & 2 & 3 & 1 & 0 & 0\\0 & -2 & -2 & -2 & 1 & 0\\0 & -2 & -3 & -3 & 0 & 1\end{bmatrix}$$

Simultaneous Gaussian Elimination

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 4 & 6 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & -2 & 1 & 0 \\ 0 & -2 & -3 & -3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1 + 2R_2]{R_3 - 2R_2} \begin{bmatrix} 1 & 0 & -1 & -3 & 2 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix}$$

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$$\xrightarrow[R_1 + {\color{red}2}R_2]{R_3 - {\color{red}2}R_2} \begin{bmatrix} 1 & 0 & -1 & -3 & 2 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1 + R_3]{R_2 + {\color{red}2}R_3} \begin{bmatrix} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & -1 & 0 & 0 & -3 & 2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -R_2} \begin{bmatrix} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix}$$

Check Your Answer

Is

$$B = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

an inverse for A?

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Check: $AB = I_3$ and $BA = I_3$

Summary

- Matrix manipulations follow from what happens to processes
- Invertible matrices correspond to invertible processes
- Invertible process means input and output determine each other