TMA4115 Matematikk 3

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Trondheim

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Lecture 11: What Is . . . The Matrix?

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Recap

- Solve linear systems by elimination
- Make systematic to ensure well-defined

Key Points

- Loose Ends
- The Language of the Matrix

Loose End from ODEs

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The shortest path between two truths in the real domain passes through the complex plane.

Hadamard

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$$\begin{aligned} c_1 e^{ix} + c_2 e^{-ix} = {} & (a_1 + ib_1)(\cos(x) + i\sin(x)) \\ & + (a_2 + ib_2)(\cos(x) - i\sin(x)) \end{aligned}$$

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Real Solutions: $y(x) = A\cos(x) + B\sin(x)$

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3. Interpret answer correctly
   - 3.1 Use “back substitution”
   - 3.2 Use “dummy variables” if have choice

GE: An Algorithm (pt 2: RREF)

```
matrix rref (matrix B)
{
  int l = 1;
  for (j=0; j<=colsize(B); j++) {
     for (i=l; i<=rowsize(B); i++) {
       if (B.i.j != 0) {
         rowswap(B,i,l);
         for (k=l+1; k<=rowsize(B); k++) {
           rowadd(B,k,l,-B.k.j/B.l.j);
         }
         l++;
         next j;
       }
     }
  }
  return B;
}
```

An Example

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$$\xrightarrow[R_3+2R_1]{R_2+2R_1} \begin{bmatrix} -1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 7 & -1 & -6 \end{bmatrix}$$

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$$\xrightarrow{R_3-7R_2} \begin{bmatrix} -1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -15 & 15 \end{bmatrix}$$

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- Full reduction: Gauss–Jordan, partial: Gauss

Loose Ends: Number of Solutions

Theorem

$$2 \Longrightarrow \infty$$

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Proof.

$y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ both solutions of

$$a_1 x_1 + \cdots + a_n x_n = b$$

$\Longrightarrow (ty_1 + sz_1), \ldots, (ty_n + sz_n)$ with $t + s = 1$ also solution:

$$\begin{aligned} &a_1(ty_1 + sz_1) + \cdots + a_n(ty_n + sz_n) \\ &\quad = t\big(a_1 y_1 + \cdots + a_n y_n\big) + s\big(a_1 z_1 + \cdots a_n z_n\big) \\ &\quad = tb + sb = b \end{aligned}$$

□

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The Rules

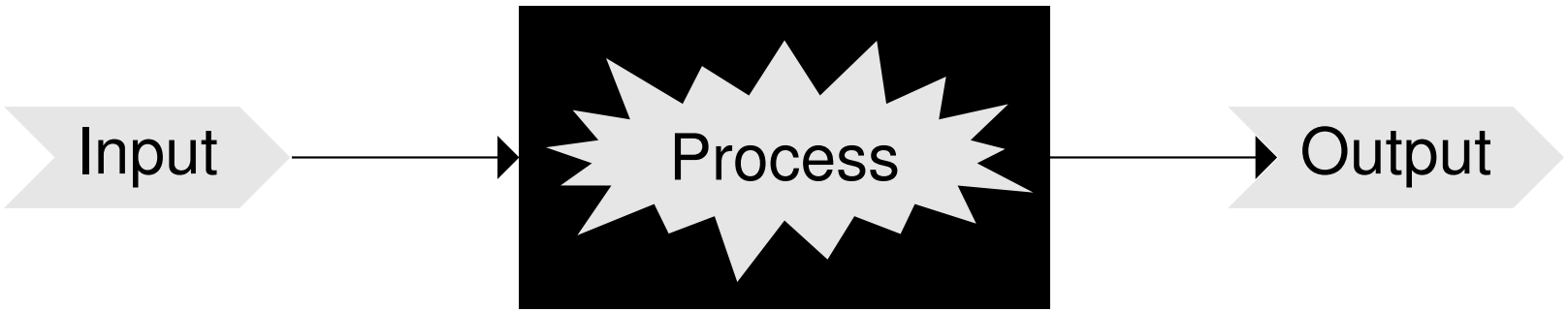
- Keep only the numbers
- Include "hidden" 0s, 1s, and -1s:

$$\text{“ ”} = 0x, \qquad x = 1x, \qquad -x = -1x$$

- Keep watch for signs

Why Linear Systems

Mathematical problems are usually processes.



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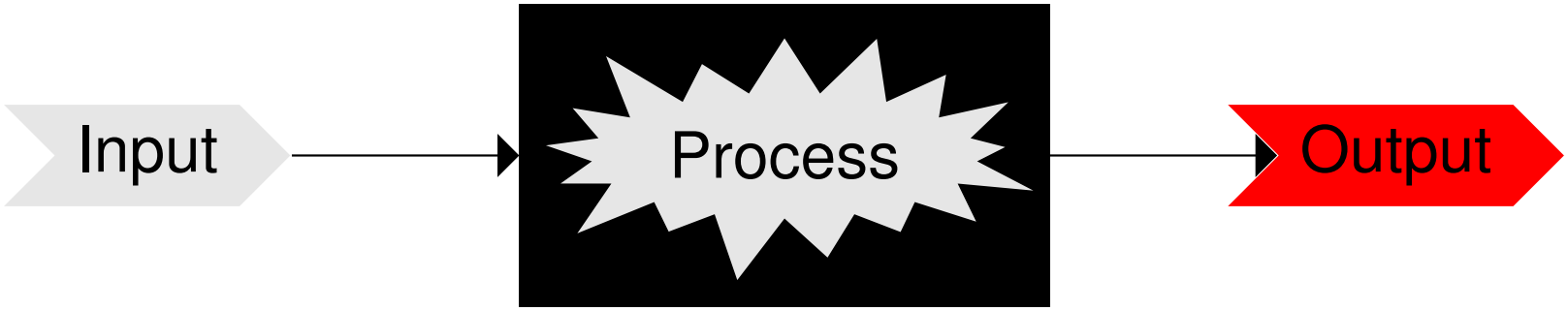

Question

1. I put in X, what do I get?

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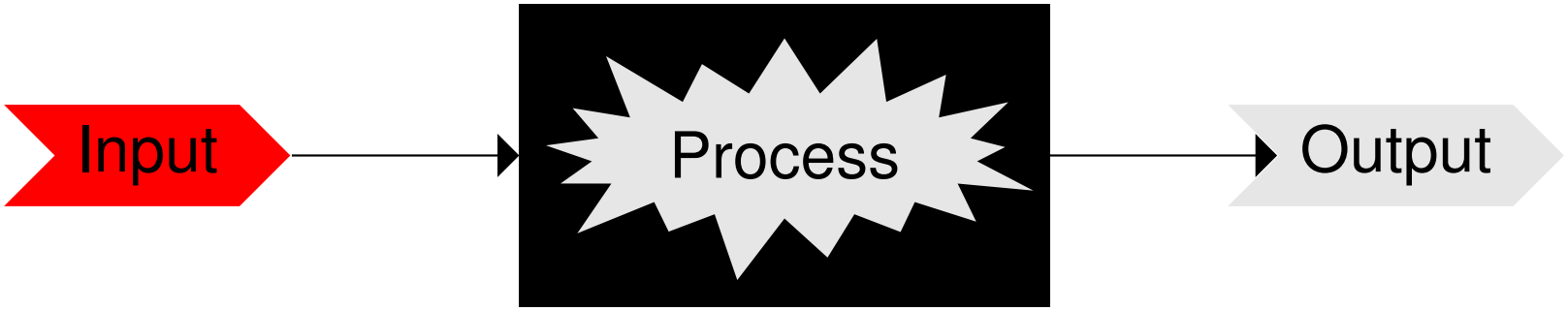

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3. I put in X, I got Y, what did I do?

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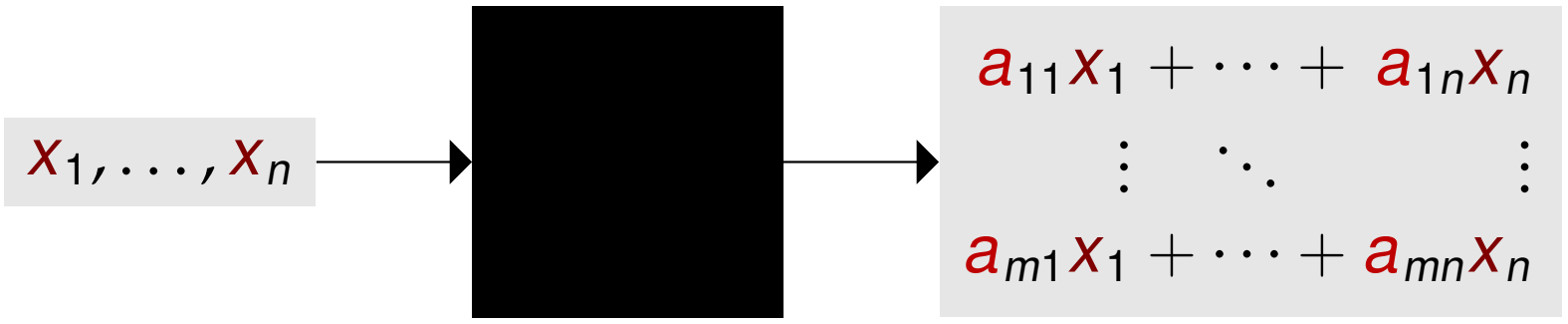
Linear systems: processes of form:



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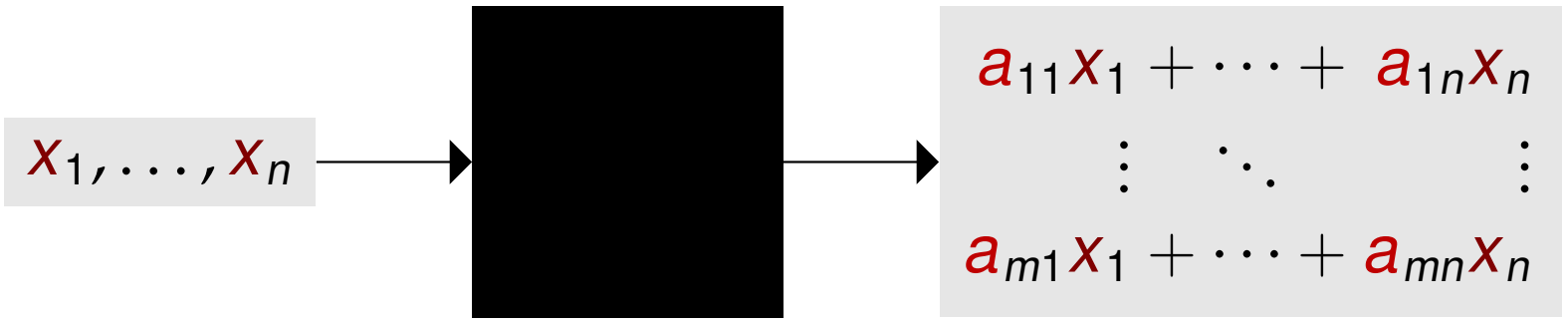

Give this linear system the label:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

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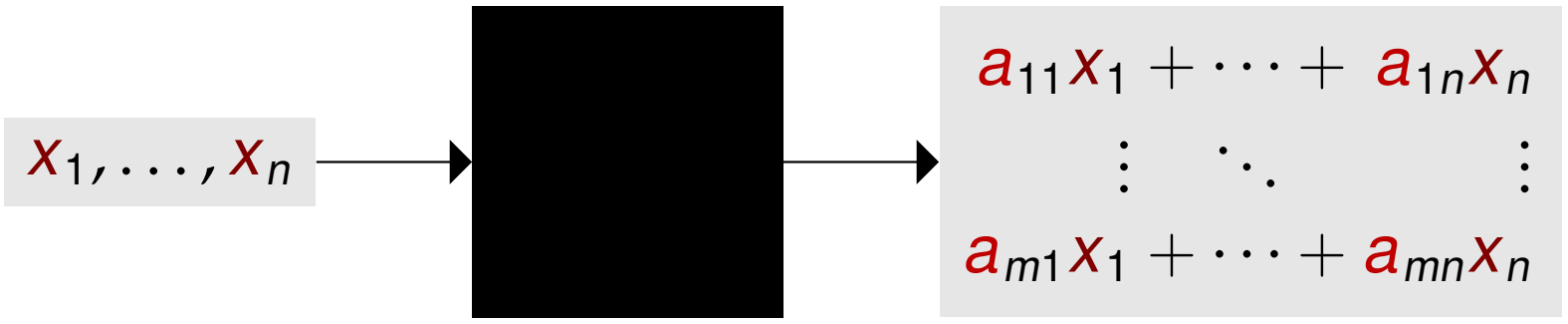

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Also note size: n inputs, m outputs: $m \times n$ matrix

Intermediate Step

What to do with processes

Compose them!

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Assume:

1. $xs \to ys$ linear system
2. $ys \to zs$ linear system

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Question

1. Is $xs \to zs$ a linear system?
2. If so, which?

Composition: Example

$$
\begin{aligned}
3x_1 + x_2 &= y_1 \\
-2x_1 + 2x_2 &= y_2 \\
4x_1 - x_2 &= y_3 \\
x_1 + x_2 &= y_4
\end{aligned}
\qquad
\begin{aligned}
3y_1 + 4y_2 + y_3 - y_4 &= z_1 \\
y_1 - 2y_2 + 2y_3 + 2y_4 &= z_2
\end{aligned}
$$

Composition: Example

$$\begin{aligned} 3x_1 + x_2 &= y_1 \\ -2x_1 + 2x_2 &= y_2 \\ 4x_1 - x_2 &= y_3 \\ x_1 + x_2 &= y_4 \end{aligned} \qquad \begin{aligned} 3y_1 + 4y_2 + y_3 - y_4 &= z_1 \\ y_1 - 2y_2 + 2y_3 + 2y_4 &= z_2 \end{aligned}$$

Compose by substitution:

$$\begin{aligned} 3y_1 + 4y_2 + y_3 - y_4 &= z_1 \\ y_1 - 2y_2 + 2y_3 + 2y_4 &= z_2 \end{aligned}$$

Composition: Example

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In Matrix–Speak

$$(x \xrightarrow{A} y)\begin{bmatrix} 3 & 1 \\ -2 & 2 \\ 4 & -1 \\ 1 & 1 \end{bmatrix} \qquad (y \xrightarrow{B} z)\begin{bmatrix} 3 & 4 & 1 & -1 \\ 1 & -2 & 2 & 2 \end{bmatrix}$$

The composition or product is

$$(x \xrightarrow{C} z)\begin{bmatrix} 4 & 9 \\ 17 & -3 \end{bmatrix}$$

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Written in this order:

$$\begin{bmatrix} 4 & 9 \\ 17 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 2 \\ 4 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 & -1 \\ 1 & -2 & 2 & 2 \end{bmatrix}$$

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- Entries of C:

$$\begin{aligned} c_{ij} &= \text{amount of } x_j \text{ in } z_i \\ &= \sum_k \Big(\text{amount of } y_k \text{ in } z_i\Big) \times \Big(\text{amount of } x_j \text{ in } y_k\Big) \\ &= \sum_k b_{ik} a_{kj} \end{aligned}$$

In Pictures

$$\begin{bmatrix} 4 & 9 \\ 17 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 & -1 \\ 1 & -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 2 \\ 4 & -1 \\ 1 & 1 \end{bmatrix}$$

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Summary

- Solve linear systems using elimination
- Linear systems model linear processes
- Composition of linear processes
  $\rightarrow$ Composition of linear systems
  $\rightarrow$ Multiplication of matrices