TMA4115 Matematikk 3

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Lecture 10: Elementary, my dear Gauss

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Recap

- Linear systems: things like

$$-x + 4y = 9$$
$$3x + 2y = 1$$

- Solve: means find numbers that make them true
- Some are easier to solve than others

Key Points

- Elementary operations change linear systems without changing solutions
- One other . . . to be revealed

Which Is Easier?

1. Solve

$$-x + 4y = 9$$
$$3x + 2y = 1$$

2. Solve

$$-x + 8 = 9$$
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Solutions are same so transform first into second

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Secondary Goal

Transform:
linear systems of 2nd type into linear systems of 1st type.

Simple Steps: Elementary Operations

1. Scaling: $\lambda \neq 0$

$$\begin{aligned} a_1x_1 + \quad a_2x_2 + \ldots + \quad a_nx_n &= \quad b \\ &\downarrow \\ \lambda a_1x_1 + \lambda a_2x_2 + \ldots + \lambda a_nx_n &= \lambda b \end{aligned}$$

2. Swapping:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\downarrow \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \end{aligned}$$

Simple Steps: Elementary Operations

3. Adding: any $\lambda \in \mathbb{R}$

$$
\begin{array}{rrrr}
a_{11}x_1 + & a_{12}x_2 + \ldots + & a_{1n}x_n = & b_1 \\
a_{21}x_1 + & a_{22}x_2 + \ldots + & a_{2n}x_n = & b_2 \\
 & \downarrow & & \\
(a_{11}+\lambda a_{21})x_1 + & (a_{12}+\lambda a_{22})x_2 + \ldots + & (a_{1n}+\lambda a_{2n})x_n = & b_1+\lambda b_2 \\
a_{21}x_1 + & a_{22}x_2 + \ldots + & a_{2n}x_n = & b_2
\end{array}
$$

Finding Solutions

What Makes It Work

In each case: if $x_1, x_2, \ldots, x_n$ satisfy 1st set, they also satisfy 2nd set.

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   then $x_1, x_2, \ldots, x_n$ are such that

$$\begin{aligned}\lambda a_1x_1 + \lambda a_2x_2 + \cdots + \lambda a_nx_n &= \lambda(a_1x_1 + a_2x_2 + \cdots + a_nx_n)\\ &= \lambda b\end{aligned}$$

Finding Solutions

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   then $x_1, x_2, \ldots, x_n$ are such that

$$\begin{aligned} (a_{11}+\lambda a_{21})x_1 + (a_{12}+\lambda a_{22})x_2 + \ldots + (a_{1n}+\lambda a_{2n})x_n &= b_1+\lambda b_2 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \end{aligned}$$

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Solution of 1st set is solution of 2nd.

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Comes from reversibility:

1. Scaling: scale by λ^{-1}
2. Swapping: swap back
3. Adding: add $-\lambda$ of 2nd to 1st

Examples

$$\begin{aligned} 2x + 3y + z &= -2 \\ 2x - 3y + 4z &= 1 \\ -x + 2y - z &= -2 \end{aligned}$$

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Stop here. Read off solution: $z = -1$, so $y = -1$, so $x = 1$

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Difference?

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Difference? No nasty fractions!

The Key Step

1. Scaling
2. Swapping
3. Adding

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Reason

It's the only one that substantially alters the system.

Project

Design a “program” to solve linear systems.

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Things to focus on:

- Specifying the problem
- Designing the algorithm
- Implementing the algorithm
- Interpreting the result
- Important Technicalities
  - Inputting the problem
  - Ensuring termination

This frame has been deliberately left blank.

Matrix

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A matrix is a rectangular grid of things.

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Key Point

Linear systems can be represented by matrices
with no loss of information

The Obligatory Matrix Stunt

$$\begin{aligned} 2x + 3y + z &= -2 \\ 2x - 3y + 4z &= 1 \\ -x + 2y - z &= -2 \end{aligned}$$

The Obligatory Matrix Stunt

$$\begin{bmatrix} 2 & 3 & 1 & -2 \\ 2 & -3 & 4 & 1 \\ -1 & 2 & -1 & -2 \end{bmatrix}$$

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$$\begin{matrix} 2 & 3 & 1 & -2 \\ 2 & -3 & 4 & 1 \\ -1 & 2 & -1 & -2 \end{matrix}$$

The Rules

- Keep only the numbers
- Include "hidden" 0s, 1s, and -1s:

$$\text{“ ”} = 0x, \qquad x = 1x, \qquad -x = -1x$$

- Keep watch for signs

Two Matrices

- Coefficient matrix: left-hand side of “=”s
- Augmented matrix: everything

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Notation:

$$\begin{bmatrix} 2 & 3 & 0 \\ 3 & -3 & 4 \\ -1 & 2 & -1 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 2 & 3 & 0 & -2 \\ 2 & -3 & 4 & 1 \\ -1 & 2 & -1 & -2 \end{array}\right]$$

Summary

- Use Gaussian elimination to solve linear systems
- Result is in echelon form
- Reduced echelon form is guaranteed unique
- Matrices encode linear systems
- Can perform GE on matrix rather than linear system