TMA4115 Matematikk 3

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Lecture 9: What's The Difference?

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Key Points

- Focus on solving problem
- Find useful description of solution set
- Use reversible techniques

Linear Systems

Linear $\Longrightarrow$ constants + multiples of x, y, z (etc.)

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$$\begin{aligned} 2x + \quad & 3y + \quad & 4z &= 7 \\ 2x^2 + \quad & 3\sqrt{y} + & 4\sin(z) &= 7 \\ \sin(2)x + & \log(3)y + \quad & e^4 z &= 7 \end{aligned}$$

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Linear system = family of linear equations.

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Key Question

What do we do with them?

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Linear system = family of linear equations.

Key Question

What do we do with them?

Answer

Solve them!

Solving Linear Systems

Solve

$$-x + 4y = 9$$
$$3x + 2y = 1$$

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$$-x + 4y = 9$$
$$3x + 2y = 1$$

Question

What does that mean?

Solving Linear Systems

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$$-x + 4y = 9$$
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Question

What does that mean?

Answer

- Find two numbers which when put in for x and y make the two statements true.
- Find all possible such pairs, or show that none exist.

ODEs and Linear Systems

Similarities

Differences

ODEs and Linear Systems

Similarities

- Focus on solving

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$\Longrightarrow$ Method subordinate to goal

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- Theoretically possible to solve exactly

ODEs and Linear Systems

Similarities

- Focus on solving

$\Longrightarrow$ Method subordinate to goal

- In practise, use approximation methods

Differences

- Theoretically possible to solve exactly

Since You Mentioned Them. . .

Linear Systems From ODEs

The final stage in solving an IVP is a linear system

Linear Systems from ODEs

Solve

$$y'' + 2y' + y = 0, \qquad y(0) = 1, y'(0) = 0$$

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- Fit to IVP:

$$ae^{-0} + b0e^{-0} = 1$$
$$-ae^{-0} - b0e^{-0} + be^{-0} = 0$$

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- Rearrange:

$$a + 0b = 1$$
$$-a + b = 0$$

Linear Systems from ODEs

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- Read off solution: $a = 1$, $b = 1$

Linear Systems from ODEs

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$$y'' + 2y' + y = 0, \qquad y(0) = 1, y'(0) = 0$$

- Solutions: e^{-x}, xe^{-x}
- General Solution: $ae^{-x} + bxe^{-x}$
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- Rearrange:

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- Read off solution: $a = 1$, $b = 1$
- Interpret correctly: solution of IVP is $e^{-x} + xe^{-x}$

Solving Linear Systems: Key Step

Solution of

$$-x + 4y = 9$$
$$3x + 2y = 1$$

Solving Linear Systems: Key Step

Solution of

$$\begin{aligned} -x + 4y &= 9 \\ 3x + 2y &= 1 \end{aligned}$$

is solution of

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Which Is Easier?

1. Solve

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$$3x + 2y = 1$$

2. Solve

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Key Idea

Solutions are same so transform first into second

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Key Idea

Solutions are same so transform first into second

Secondary Goal

Transform:
linear systems of 2nd type into linear systems of 1st type.

Simple Steps: Elementary Operations

1. Scaling: $\lambda \neq 0$

$$\begin{aligned} a_1x_1 + \quad a_2x_2 + \ldots + \quad a_nx_n &= \quad b \\ &\downarrow \\ \lambda a_1x_1 + \lambda a_2x_2 + \ldots + \lambda a_nx_n &= \lambda b \end{aligned}$$

2. Swapping:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\downarrow \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \end{aligned}$$

Simple Steps: Elementary Operations

3. Adding: any $\lambda \in \mathbb{R}$

$$
\begin{array}{rrrr}
a_{11}x_1 + & a_{12}x_2 + \ldots + & a_{1n}x_n = & b_1 \\
a_{21}x_1 + & a_{22}x_2 + \ldots + & a_{2n}x_n = & b_2 \\
 & \downarrow & & \\
(a_{11}+\lambda a_{21})x_1 + & (a_{12}+\lambda a_{22})x_2 + \ldots + & (a_{1n}+\lambda a_{2n})x_n = & b_1+\lambda b_2 \\
a_{21}x_1 + & a_{22}x_2 + \ldots + & a_{2n}x_n = & b_2
\end{array}
$$

Finding Solutions

- In each case: if $x_1, x_2, \ldots, x_n$ satisfy 1st set, they also satisfy 2nd set.

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"$\Longleftrightarrow$" comes from reversibility

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"$\Longleftrightarrow$" comes from reversibility

Each step can be reversed by a similar step.

Examples

$$
\begin{aligned}
2x + 3y + z &= -2 \\
2x - 3y + 4z &= 1 \\
-x + 2y - z &= -2
\end{aligned}
$$

Examples

$$\begin{array}{rrrr} 2x+ & 3y+ & z= & -2 \\ 2x- & 3y+ & 4z= & 1 \\ -x+ & 2y- & z= & -2 \end{array} \xrightarrow{R_2-R_1} \begin{array}{rrrr} 2x+ & 3y+ & z= & -2 \\ & -6y+ & 3z= & 3 \\ -x+ & 2y- & z= & -2 \end{array}$$

Examples

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$$\xrightarrow{R_3+{}^1\!/_2 R_1} \begin{array}{rrrr} 2x+ & 3y+ & z= & -2 \\ & -6y+ & 3z= & 3 \\ & 3{}^1\!/_2 y- & {}^1\!/_2 z= & -3 \end{array}$$

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$$\xrightarrow{R_3 + {}^1\!/_2 R_1} \begin{aligned} 2x + 3y + z &= -2 \\ -6y + 3z &= 3 \\ 3{}^1\!/_2 y - {}^1\!/_2 z &= -3 \end{aligned}$$

$$\xrightarrow{R_3 + {}^7\!/_{12} R_2} \begin{aligned} 2x + 3y + z &= -2 \\ -6y + 3z &= 3 \\ 1{}^1\!/_4 z &= -1{}^1\!/_4 \end{aligned}$$

Examples

$$\begin{array}{rrrr} 2x+3y+ & z=-2 \\ 2x-3y+4z= & 1 \\ -x+2y- & z=-2 \end{array} \xrightarrow{R_2-R_1} \begin{array}{rrrr} 2x+ & 3y+ & z=-2 \\ & -6y+3z= & 3 \\ -x+ & 2y- & z=-2 \end{array}$$

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Stop here. Read off solution: $z=-1$, so $y=-1$, so $x=1$

Slightly Simpler

$$\begin{aligned} 2x + 3y + z &= -2 \\ 2x - 3y + 4z &= 1 \\ -x + 2y - z &= -2 \end{aligned}$$

Slightly Simpler

$$\begin{array}{l} 2x + 3y + z = -2 \\ 2x - 3y + 4z = 1 \\ -x + 2y - z = -2 \end{array} \xrightarrow[R_3+2R_1]{R_1 \leftrightarrow R_3} \begin{array}{l} -x + 2y - z = -2 \\ 2x - 3y + 4z = 1 \\ 2x + 3y + z = -2 \end{array}$$

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$$\begin{array}{l} 2x+3y+\ \ z=-2 \\ 2x-3y+4z=\ \ 1 \\ -x+2y-\ \ z=-2 \end{array} \xrightarrow[R_3+2R_1]{R_1\leftrightarrow R_3} \begin{array}{l} -x+2y-\ \ z=-2 \\ 2x-3y+4z=\ \ 1 \\ 2x+3y+\ \ z=-2 \end{array}$$

$$\xrightarrow[R_3+2R_1]{R_2+2R_1} \begin{array}{l} -x+2y-\ \ z=-2 \\ \ \ \ \ y+2z=-3 \\ \ \ \ 7y-\ \ z=-6 \end{array}$$

Slightly Simpler

$$\begin{array}{rcl} 2x+3y+z=-2 & & -x+2y-z=-2 \\ 2x-3y+4z=1 & \xrightarrow[R_3+2R_1]{R_1\leftrightarrow R_3} & 2x-3y+4z=1 \\ -x+2y-z=-2 & & 2x+3y+z=-2 \end{array}$$

$$\begin{array}{rcl} & & -x+2y-z=-2 \\ & \xrightarrow[R_3+2R_1]{R_2+2R_1} & y+2z=-3 \\ & & 7y-z=-6 \end{array}$$

$$\begin{array}{rcl} & & -x+2y-z=-2 \\ & \xrightarrow{R_3+7R_2} & y+2z=-3 \\ & & -15z=15 \end{array}$$

Slightly Simpler

$$\begin{array}{rcl} 2x+3y+z=-2 & & -x+2y-z=-2 \\ 2x-3y+4z=1 & \xrightarrow[R_3+2R_1]{R_1\leftrightarrow R_3} & 2x-3y+4z=1 \\ -x+2y-z=-2 & & 2x+3y+z=-2 \end{array}$$

$$\begin{array}{rcl} & & -x+2y-z=-2 \\ & \xrightarrow[R_3+2R_1]{R_2+2R_1} & y+2z=-3 \\ & & 7y-z=-6 \end{array}$$

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Difference?

Slightly Simpler

$$\begin{array}{rrrr} 2x+3y+ & z & = & -2 \\ 2x-3y+4z & & = & 1 \\ -x+2y- & z & = & -2 \end{array} \xrightarrow[R_3+2R_1]{R_1 \leftrightarrow R_3} \begin{array}{rrrr} -x+2y- & z & = & -2 \\ 2x-3y+4z & & = & 1 \\ 2x+3y+ & z & = & -2 \end{array}$$

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$$\xrightarrow{R_3+7R_2} \begin{array}{rrrr} -x+2y- & z & = & -2 \\ y+ & 2z & = & -3 \\ & -15z & = & 15 \end{array}$$

Difference? No nasty fractions!

Summary

- Solving linear systems means finding numbers that make them true
- Some linear systems are easier than others
- Elementary operations change linear systems without changing solutions
- Choose to make simpler