TMA4115 Matematikk 3

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

Spring 2010

Lecture 8: Ode to ODE

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5th February 2010

Recap

- Solve inhomogeneous by:
  1. Solve homogeneous
  2. Find particular solution
  3. Solve IVP
- Constant coefficients + periodic forcing displays . . . unusual behaviour

Key Points

- Use solutions of homogeneous to “simplify” search for solution of inhomogeneous

Lessons From Pendula

Mathematical Models are

1. Idealised

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1. Error in Implementation

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Fix it, and hope no-one notices.

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   Find boundaries in model.

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   Fix it, and hope no-one notices.
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   Find boundaries in model.
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   Publish and get a Nobel prize!

Form(ula) versus Function

Definition

A function $f : I \to \mathbb{R}$ *is a*

rule

assigning a real number to each number in I*.*

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1. $t \mapsto t^2$
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3. $\theta \mapsto x$–coordinate of point with angle θ
4. $t \mapsto t$–value of solution of

$$y'' + p(x)y' + q(x)y = r(x)$$

Unknown Knowns

Key Point

- $f(t)$ must be uniquely determined
- $f(t)$ need not be known

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Example

1. "$t \mapsto t$–value of solution of ODE" is fine
2. $2\sin(x)\cos(x)$ and $\sin(2x)$ are same function

Form(ula) versus Function

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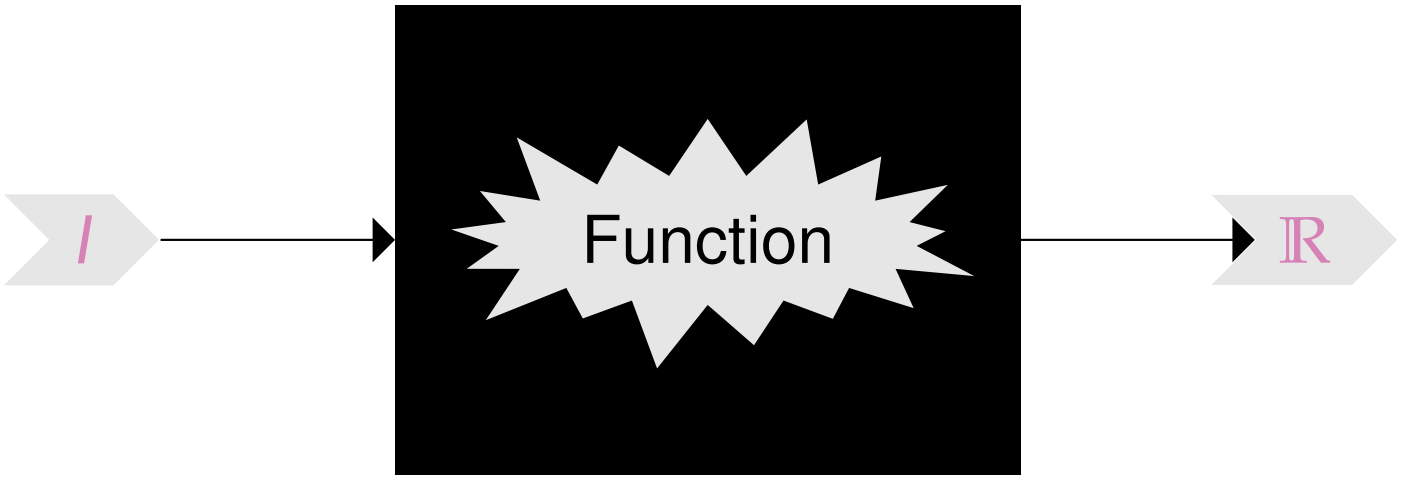

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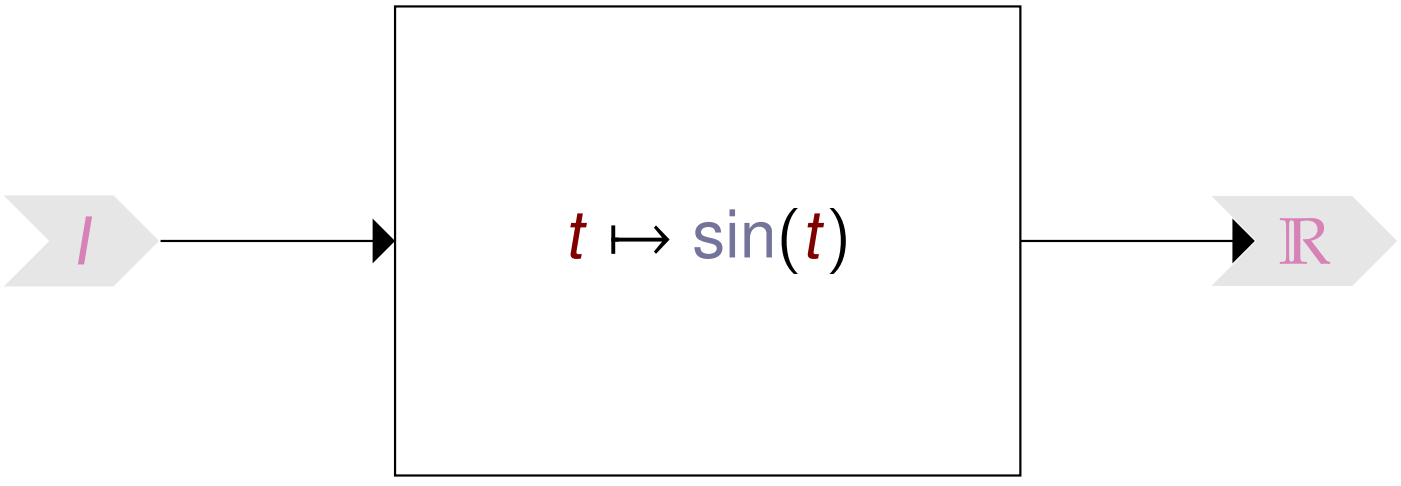

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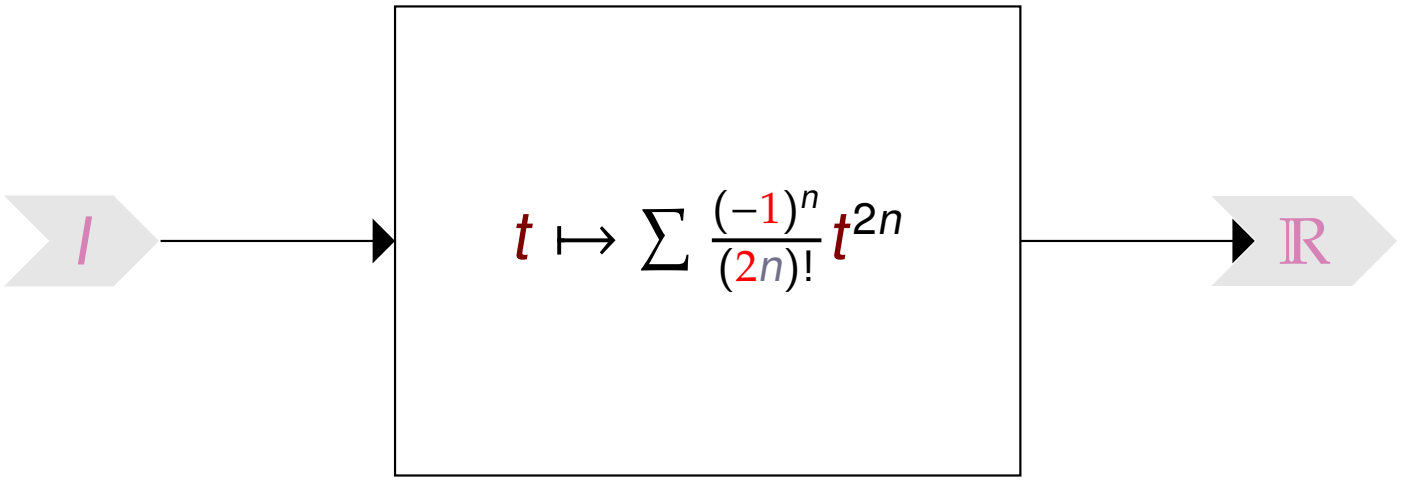

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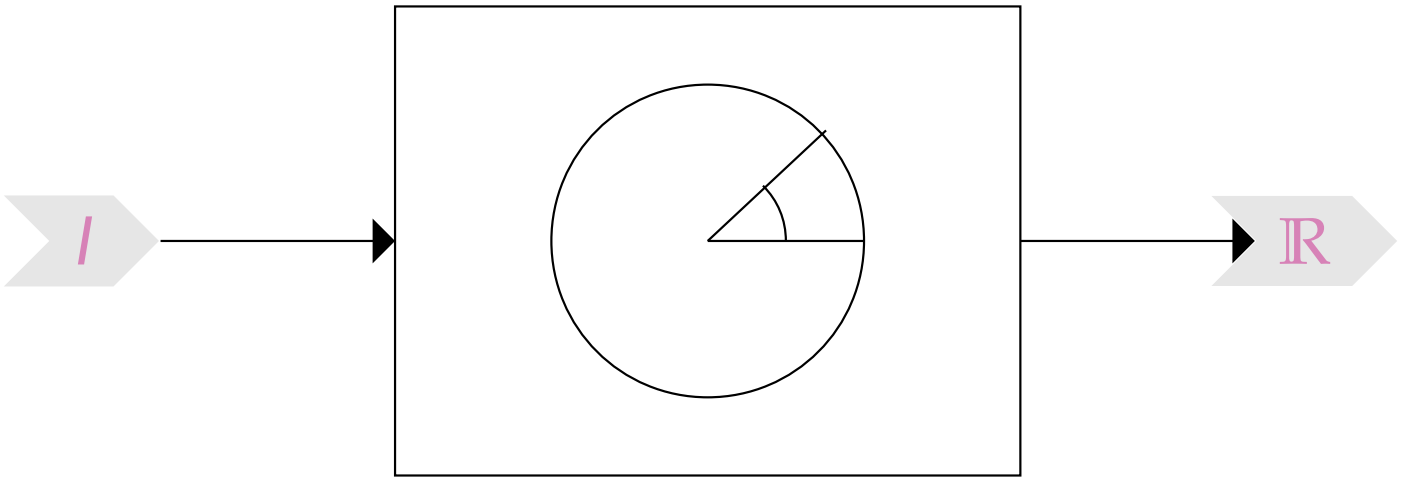

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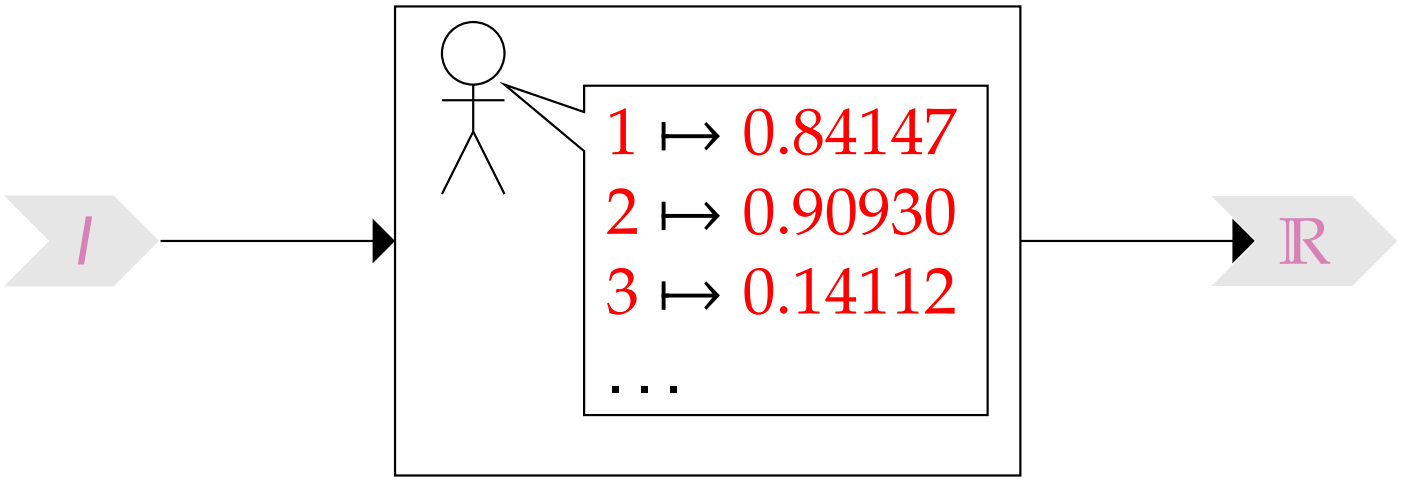

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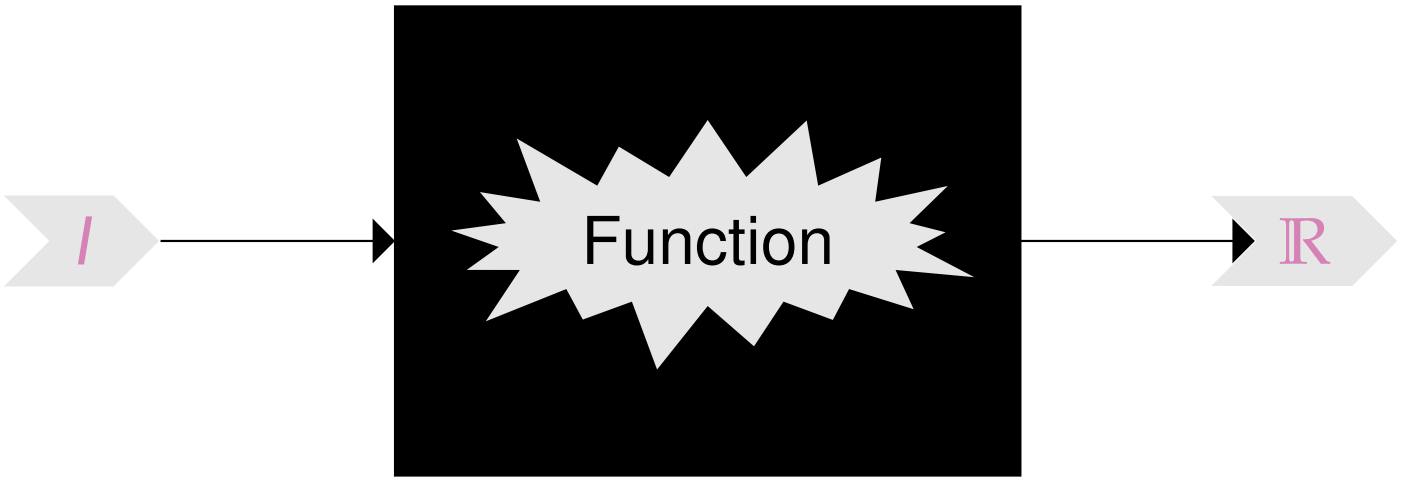
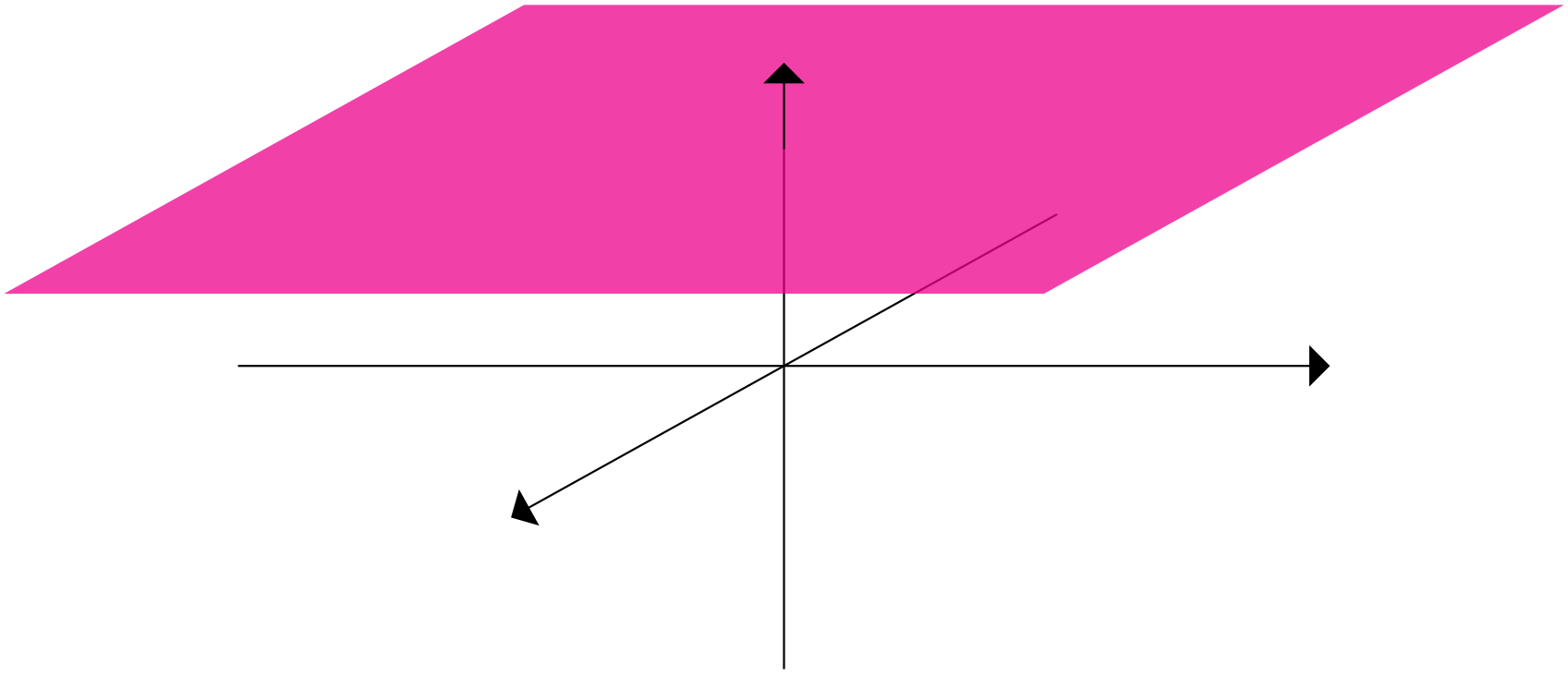


III : Implementation Is Irrelevant

Finding a Plane: the ODE Way



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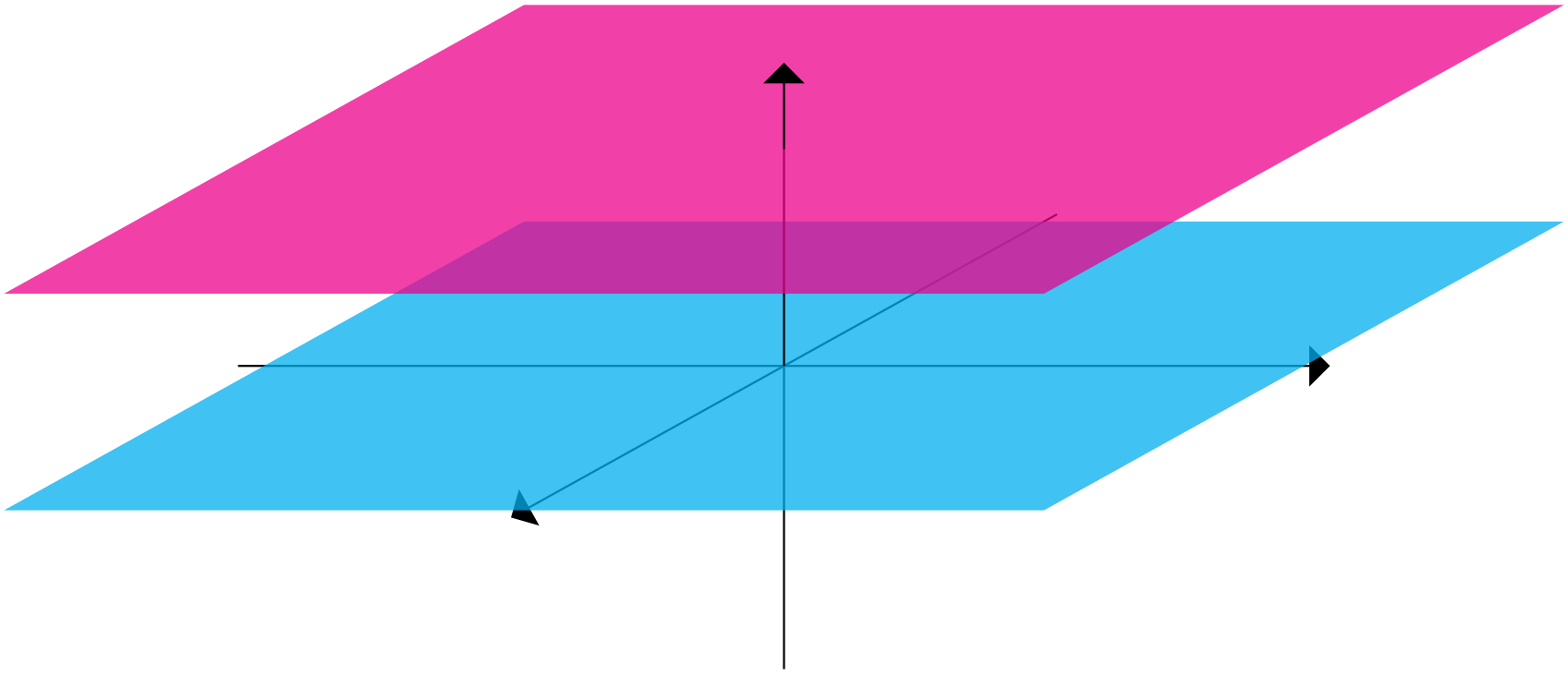
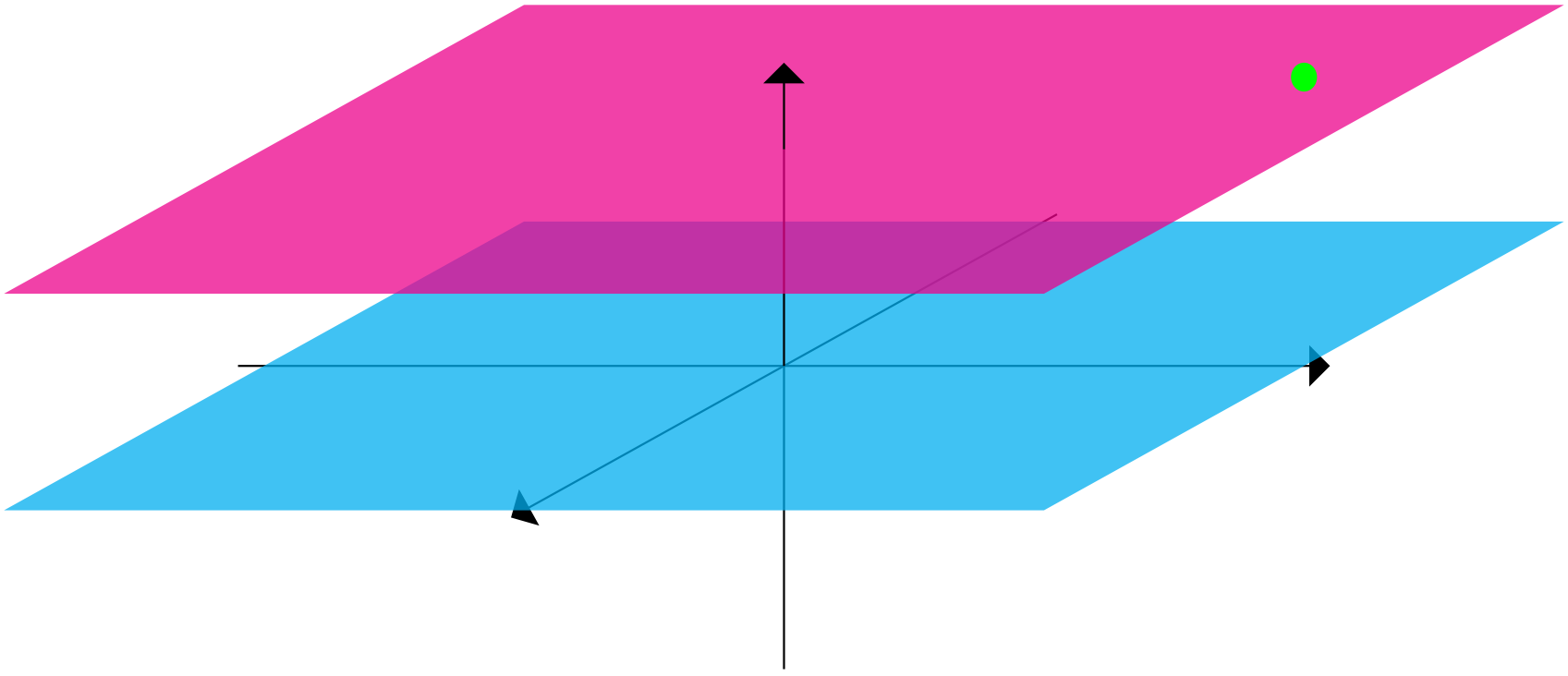


1. Find plane through origin.

Finding a Plane: the ODE Way



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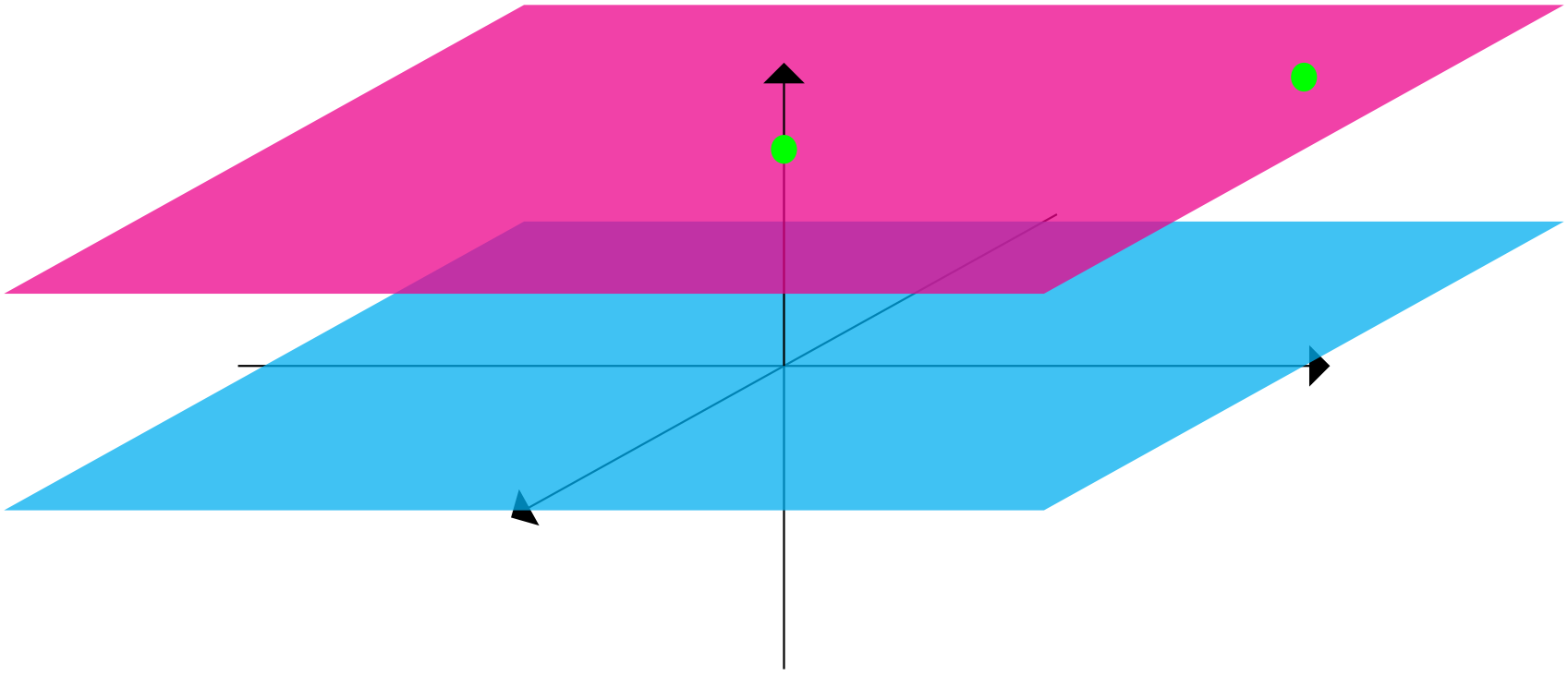

1. Find plane through origin.
2. Find point on plane.
3. Simplify search: look in “obvious” direction.

Spot of Geometry

- Plane through origin: $\{\lambda\vec{v} + \mu\vec{w}\}$
- Translated plane: $\{\vec{u} + \lambda\vec{v} + \mu\vec{w}\}$

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$$\vec{u}^{\mathsf{T}}\vec{v} = 0, \qquad \vec{u}^{\mathsf{T}}\vec{w} = 0$$

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- $\Big($ General nonsense $\Longrightarrow$ no loss of generality $\Big)$

Simplifying Problems

$$y'' + y = \tan\left(\log\left(\frac{\sin(x)}{2 + \cos(x)}\right)\right)$$

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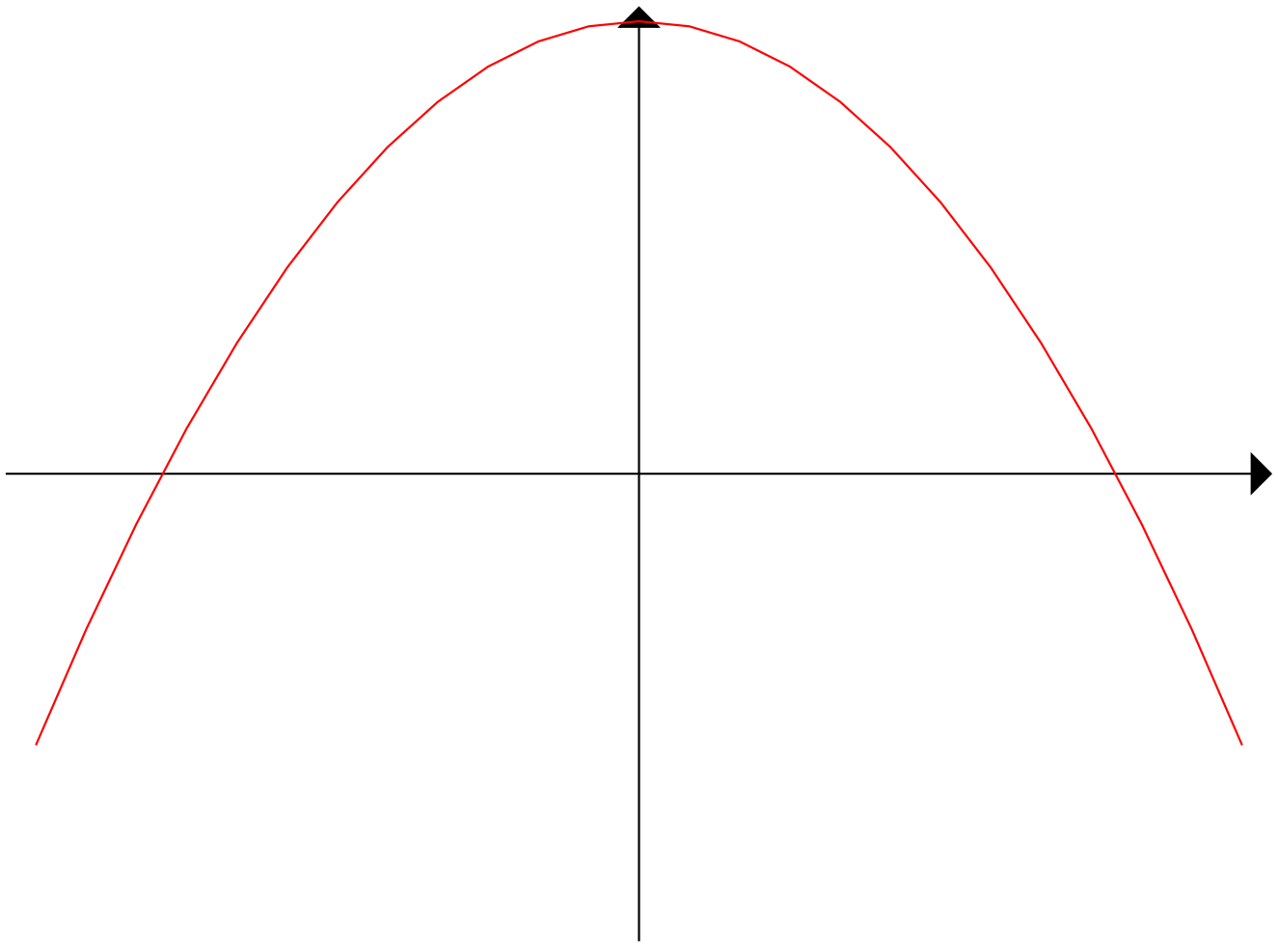
Solution of Inhomogeneous? $y(x)$?

- $y(x)$ fixed by $y(x_0)$, $y'(x_0)$, any $x_0 \in I$.
- Have solution of homogeneous ODE through the same points:

$$c_{x_0} \sin(x) + d_{x_0} \cos(x)$$

In Pictures



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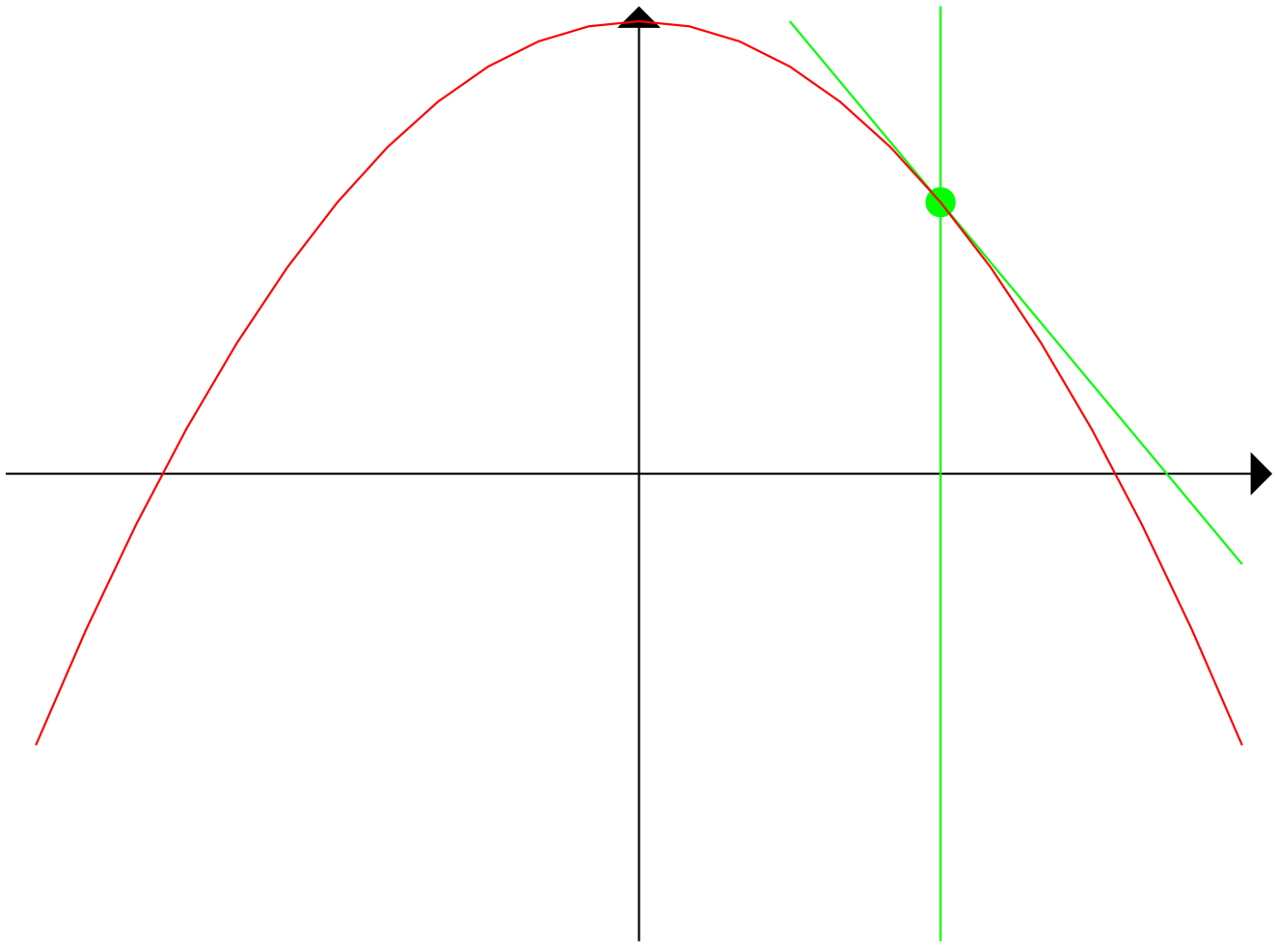

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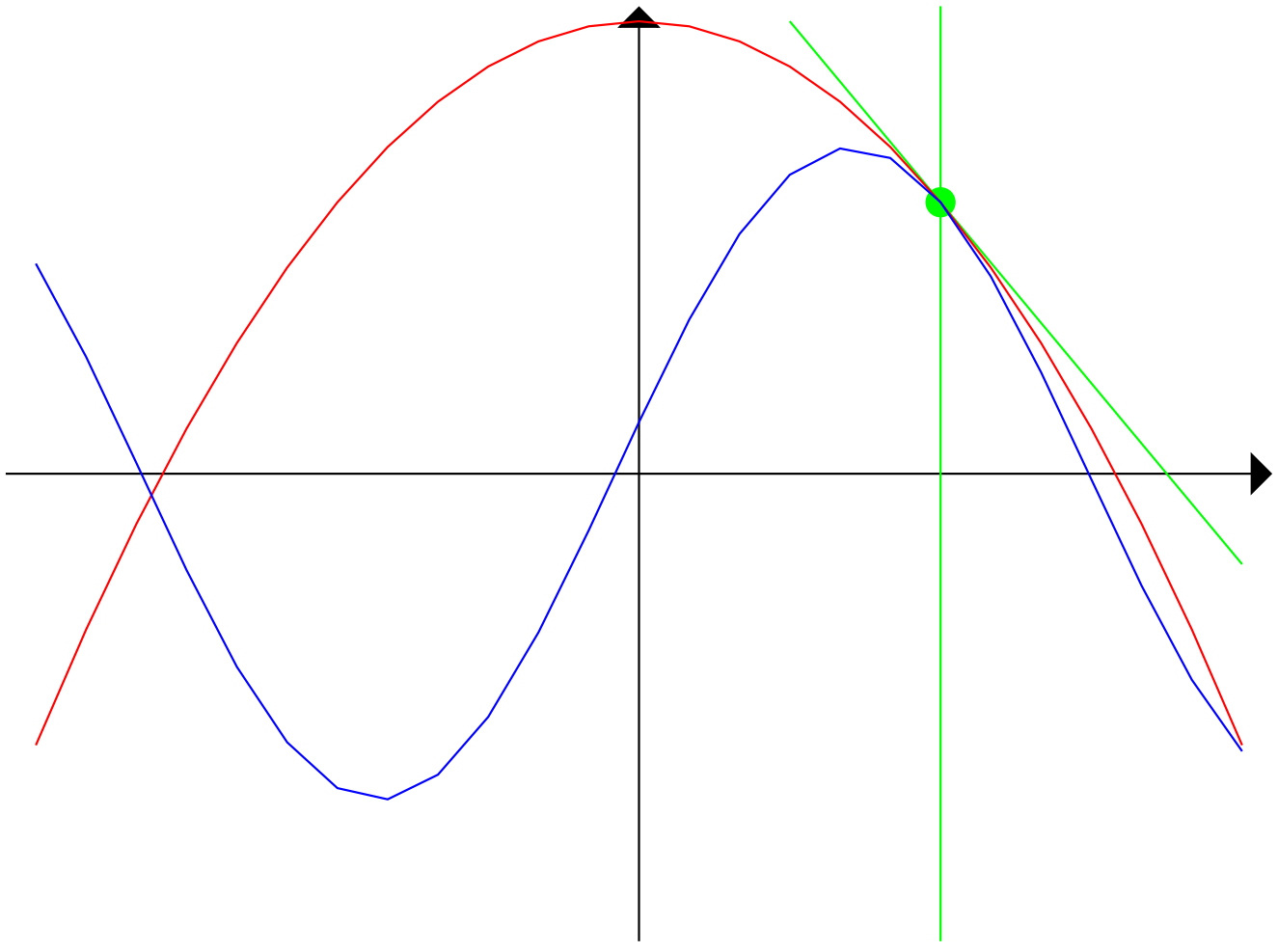

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Key Principle

Approximate unknown by most useful knowns.

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Approximate unknown by most useful knowns.

- In this case, solutions of homogeneous ODE.
- Computation: polynomials
- Signal analysis: trigonomials
- Plotting: piecewise-linear
- ...

What's the Point?

$$y(x) = c_{x_0} \sin(x) + d_{x_0} \cos(x)$$

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- $x_0 \mapsto c_{x_0}$ and $x_0 \mapsto d_{x_0}$ are functions!

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- $x_0 \mapsto c_{x_0}$ and $x_0 \mapsto d_{x_0}$ are functions!

Conclusion

Have functions $c(x)$, $d(x)$ such that for all $x \in I$

$$y(x) = c(x)y_1(x) + d(x)y_2(x)$$
$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x)$$

So What?

Know $y(x)$

$\Longleftrightarrow$

Know $c(x)$ and $d(x)$

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Bad Trade?

1 unknown function $\longleftrightarrow$ 2 unknown functions

So What?

Know $y(x)$

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Know $c(x)$ and $d(x)$

Bad Trade?

1 unknown function $\longleftrightarrow$ 2 unknown functions

Maybe we gain elsewhere. . .

The Trade

Conditions:

$$y(x) = c(x)y_1(x) + d(x)y_2(x) \tag{1}$$
$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x) \tag{2}$$

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- But (2) is not the derivative of (1)!

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Differentiating:

$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x) + c'(x)y_1(x) + d'(x)y_2(x)$$

The Trade

Conditions:

$$y(x) = c(x)y_1(x) + d(x)y_2(x) \quad (1)$$
$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x) \quad (2)$$

- But (2) is not the derivative of (1)!

Differentiating:

$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x) + c'(x)y_1(x) + d'(x)y_2(x)$$

Thus:

$$0 = c'(x)y_1(x) + d'(x)y_2(x)$$

Use The ODE, Luke

$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x)$$

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Differentiating Again:

$$y''(x) = c(x)y_1''(x) + d(x)y_2''(x) + c'(x)y_1'(x) + d'(x)y_2'(x)$$

Use The ODE, Luke

$$y'(x) = c(x)y_1'(x) + d(x)y_2'(x)$$

Differentiating Again:

$$y''(x) = c(x)y_1''(x) + d(x)y_2''(x) + c'(x)y_1'(x) + d'(x)y_2'(x)$$

Substituting In:

$$y'' + p(x)y' + q(x)y = r(x)$$

$$\begin{aligned} c(x)y_1''(x) + d(x)y_2''(x) + c'(x)y_1'(x) + d'(x)y_2'(x) \\ + p(x)\Big(c(x)y_1'(x) + d(x)y_2'(x)\Big) \\ + q(x)\Big(c(x)y_1(x) + d(x)y_2(x)\Big) = r(x) \end{aligned}$$

Rearranging

$$
\begin{aligned}
y'' + p(x)y' + q(x)y &= r(x) \\
cy_1'' + dy_2'' + c'y_1' + d'y_2' & \\
+p\left(cy_1' + dy_2'\right) & \\
+q\left(cy_1 + dy_2\right) &= r
\end{aligned}
$$

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+p\left(cy_1' + dy_2'\right) & \\
+q\left(cy_1 + dy_2\right) &= r \\
c\left(y_1'' + py_1' + qy_1\right) & \\
+ d\left(y_2'' + py_2' + qy_2\right) & \\
+ c'y_1' + d'y_2' &= r
\end{aligned}
$$

Rearranging

$$y'' + p(x)y' + q(x)y = r(x)$$

$$\begin{aligned} cy_1'' + dy_2'' + c'y_1' + d'y_2' \\ +p\Big(cy_1' + dy_2'\Big) \\ +q\Big(cy_1 + dy_2\Big) = r \end{aligned}$$

$$c'y_1' + d'y_2' = r$$

Rearranging

$$y'' + p(x)y' + q(x)y = r(x)$$

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$$\begin{aligned} c'y_1' + d'y_2' &= r \\ c'y_1 + d'y_2 &= 0 \end{aligned}$$

Solving

$$\begin{aligned} c'y'_1 &+ d'y'_2 &&= r \\ c'y_1 &+ d'y_2 &&= 0 \end{aligned}$$

Solving

$$c' y_1' y_1 + d' y_2' y_1 = r y_1$$
$$c' y_1 y_1' + d' y_2 y_1' = 0$$

Solving

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\begin{aligned}
c' y_1' y_1 + d' y_2' y_1 &= r y_1 \\
c' y_1 y_1' + d' y_2 y_1' &= 0 \\
\hline
d' \left(y_1 y_2' - y_1' y_2 \right) &= r y_1
\end{aligned}
$$

Solving

$$
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$$

$$
d' W(y_1, y_2) = \quad r y_1
$$

Solving

$$\begin{aligned} c'y_1'y_1 + d'y_2'y_1 &= ry_1 \\ c'y_1y_1' + d'y_2y_1' &= 0 \end{aligned}$$

$$\overline{d'\left(y_1y_2' - y_1'y_2\right) = ry_1}$$

$$d' = \frac{ry_1}{W(y_1, y_2)}$$

Solving

$$\begin{aligned} c'y_1'y_1 + d'y_2'y_1 &= ry_1 \\ c'y_1y_1' + d'y_2y_1' &= 0 \end{aligned}$$

$$\overline{d'\left(y_1y_2' - y_1'y_2\right) = ry_1}$$

$$d = \int \frac{ry_1}{W(y_1, y_2)} \mathrm{d}x$$

Solving

$$\begin{aligned} c' y_1' y_1 + d' y_2' y_1 &= r y_1 \\ c' y_1 y_1' + d' y_2 y_1' &= 0 \end{aligned}$$

$$d' \left(y_1 y_2' - y_1' y_2 \right) = r y_1$$

$$d = \int \frac{r y_1}{W(y_1, y_2)} \mathrm{d}x$$

$$c = -\int \frac{r y_2}{W(y_1, y_2)} \mathrm{d}x$$

Variation of Parameters

Particular Solution

$$y(x) = c(x)y_1(x) + d(x)y_2(x)$$

$$c(x) = -\int^x \frac{r(t)y_2(t)}{W(y_1, y_2)(t)}\mathrm{d}t$$

$$d(x) = \int^x \frac{r(t)y_1(t)}{W(y_1, y_2)(t)}\mathrm{d}t$$

Example

$$y'' - 2y' + y = e^x \sin(x)$$

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Wronskian: $W(y_1, y_2) = e^x(e^x + xe^x) - e^x xe^x = e^{2x}$

Example

$$y'' - 2y' + y = e^x \sin(x)$$

Homogeneous: e^x, xe^x
Wronskian: $W(y_1, y_2) = e^x(e^x + xe^x) - e^x xe^x = e^{2x}$
Variation of Parameters:

$$\begin{aligned} c(x) &= -\int^x \frac{e^t \sin(t) t e^t}{e^{2t}} dt \\ &= -\int^x t \sin(t) dt \\ &= [t \cos(t)]^x - \int^x \cos(t) dt \\ &= x \cos(x) - \sin(x) \end{aligned}$$

Example ctd.

$$
\begin{aligned}
d(x) &= \int^{x} \frac{e^{t} \sin(t) e^{t}}{e^{2t}} \mathrm{d}t \\
&= \int^{x} \sin(t) \mathrm{d}t \\
&= -\cos(x)
\end{aligned}
$$

Example ctd.

$$
\begin{aligned}
d(x) &= \int^{x} \frac{e^{t}\sin(t)e^{t}}{e^{2t}}\,dt \\
&= \int^{x} \sin(t)\,dt \\
&= -\cos(x)
\end{aligned}
$$

General Solution:

$$
\begin{aligned}
\Big(x\cos(x) - \sin(x)\Big)e^{x} - \cos(x)xe^{x} + ae^{x} + bxe^{x} \\
= -\sin(x)e^{x} + ae^{x} + bxe^{x}
\end{aligned}
$$

Example ctd.

$$
\begin{aligned}
d(x) &= \int^{x} \frac{e^{t}\sin(t)e^{t}}{e^{2t}}\,dt \\
&= \int^{x} \sin(t)\,dt \\
&= -\cos(x)
\end{aligned}
$$

General Solution:

$$
\Big(x\cos(x) - \sin(x)\Big)e^{x} - \cos(x)xe^{x} + ae^{x} + bxe^{x}
$$

$$
= -\sin(x)e^{x} + ae^{x} + bxe^{x}
$$

Always check!

Example

$$x^2 y'' - 2x y' + 2y = x^3 \cos(x)$$

Example

$$y'' - 2x^{-1}y' + 2x^{-2}y = x\cos(x)$$

Example

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Homogeneous: x, x^2

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Example

$$y'' - 2x^{-1}y' + 2x^{-2}y = x\cos(x)$$

Homogeneous: x, x^2
Wronskian: $W(y_1, y_2) = x2x - x^2 = x^2$
Variation of Parameters:

$$\begin{aligned} c(x) &= -\int^x \frac{t\cos(t)t^2}{t^2}\mathrm{d}t \\ &= -\int^x t\cos(t)\mathrm{d}t \\ &= -x\sin(x) - \cos(x) \end{aligned}$$

Example ctd

$$
\begin{aligned}
d(x) &= \int^{x} \frac{t\cos(t)t}{t^2}\mathrm{d}t \\
&= \int^{x} \cos(t)\mathrm{d}t \\
&= \sin(x)
\end{aligned}
$$

Example ctd

$$
\begin{aligned}
d(x) &= \int^{x} \frac{t\cos(t)t}{t^2}\mathrm{d}t \\
&= \int^{x} \cos(t)\mathrm{d}t \\
&= \sin(x)
\end{aligned}
$$

General Solution:

$$
\begin{aligned}
-\Big(x\sin(x)+\cos(x)\Big)x + x^2\sin(x) + ax + bx^2 \\
= -x\cos(x) + ax + bx^2
\end{aligned}
$$

Summary

- ODEs have solutions
- No guaranteed method for finding
- Techniques for special cases
- Use with care!
- Use partial information to get more