TMA4115 Matematikk 3

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Lecture 6: The Grand Tour Continues

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Recap

Uniqueness

$$y'' + p(x)y' + q(x)y = 0$$
$$y(x_0) = K_0,\ y'(x_0) = K_1$$

Interval I:

1. $p(x)$, $q(x)$ continuous on I
2. $x_0 \in I$

Solution exists and is unique

Key Points

- Wronskian:

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

- Use Wronskian to test for linear independence
- Inhomogenous ODEs:

$$y'' + p(x)y' + q(x)y = r(x)$$

- Solve by intelligent guesswork

Testing Linear Independence

Tests

Solutions y_1 and y_2

1. Is $y_1 = 0$?
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
3. Is $y_1'(x_0) \neq 0$ and $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1$?

Yes to any $\Longrightarrow$ dependent No to all $\Longrightarrow$ independent

Examples

$$y'' - 2y' + y = 0$$

Solutions: e^x,

Examples

$$y'' - 2y' + y = 0$$

Solutions: e^x, xe^x

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Solutions: $y_2(x) = e^x$, $y_1(x) = xe^x$

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Question: Are these linearly independent?

1. Is $y_1 = 0$?
   No. $2e^2 \neq 0$.

Examples

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   E.g. $x_0 = 0$: "No

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   ($x_0 = 0$): "Yes ($0e^0 + e^0 \neq 0$) and No ($e^x \neq xe^x$)".

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Conclusion: Linearly independent

Examples

$$y'' - 3x^{-1}y' + 3x^{-2}y = 0$$

Solutions: $y_1(x) = x$, $y_2(x) = x^3$
Question: Are these linearly independent?

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Conclusion: Linearly independent

Examples

$$y'' + 4y = 0$$

Solutions: $y_1(x) = \sin(2x)$, $y_2(x) = \sin(x)\cos(x)$
Question: Are these linearly independent?

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   E.g. ($x_0 = 0$): "Yes ($2 \neq 0$) and Yes ($\sin(x)\cos(x) = 2/4 \sin(2x)$)".

Conclusion: Not linearly independent
(aka linearly dependent)

Testing Linear Independence

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2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?

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   $y_3 := y_2 - \frac{y_2(x_0)}{y_1(x_0)} y_1$ solution with $y_3(x_0) = 0$

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   $y_3 = 0 \iff y_3'(x_0) = 0$
   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y_2'(x_0) = \frac{y_2(x_0)}{y_1(x_0)} y_1'(x_0)$

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   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y_2'(x_0) y_1(x_0) - y_2(x_0) y_1'(x_0) = 0$
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   $y_3 := y_2 - \frac{y_2'(x_0)}{y_1'(x_0)} y_1$ solution with $y_3'(x_0) = 0$
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   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y_2(x_0) y_1'(x_0) - y_2'(x_0) y_1(x_0) = 0$

Testing Linear Independence

1. Is $y_1 = 0$?
   Are $y_1(x_0) = 0$ and $y_1'(x_0) = 0$?
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
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Claim

y_1 and y_2 not linearly independent

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1. $y_1(x_0) = 0$ and $y_1'(x_0) = 0 \implies$ test 1 holds

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3. $y_1(x_0) \neq 0 \implies$ test 3 holds

Introducing . . . The Wronskian

Definition

The Wronskian of two functions

y_1, y_2

is

$$W(y_1, y_2) := y_1 y_2' - y_1' y_2$$

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1. $W(x, x^2) = x2x - x^2 = x^2$
2. $W(e^x, e^{-x}) = e^x(-e^{-x}) - e^x e^{-x} = -2$
3. $W(\cos(x), \sin(x)) =$
   $\cos(x)\cos(x) - (-\sin(x)\sin(x)) = 1$

Wronskian and Linear Independence

$$y'' + p(x)y' + q(x)y = 0$$

y_1 and y_2 solutions of ODE.

y_1, y_2 linearly independent

$$\Longleftrightarrow$$

$W(y_1, y_2)(x) \neq 0$ for all x

$$\Longleftrightarrow$$

$W(y_1, y_2)(x_0) \neq 0$ for some x_0

The Norwegian Connection

Really Weird Fact

$W(y_1, y_2)$ depends on ODE and almost not on y_1 and y_2!

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$$\begin{aligned} W(a_1x + b_1x^2, a_2x + b_2x^2) &= (a_1x + b_1x^2)(a_2 + 2b_2x) \\ &\quad - (a_1 + 2b_1x)(a_2x + b_2x^2) \end{aligned}$$

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Always a multiple of x^2!

Abel's Identity

$$
\begin{aligned}
W(y_1, y_2)' &= y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' \\
&= y_1(-p y_2' - q y_2) - (-p y_1' - q y_1) y_2 \\
&= -p(y_1 y_2' - y_1' y_2) \\
&= -p W(y_1, y_2)
\end{aligned}
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1st Order Linear ODE!

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1st Order Linear ODE!
Solution:

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0) e^{-\int_{x_0}^{x} p(t)\mathrm{d}t}$$

Caution

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But x, $\sin(x)$ are linearly independent.

The Next Step: Inhomogeneous ODEs

Introducing . . .

$$y'' + p(x)y' + q(x)y = r(x)$$

2nd Order Inhomogeneous Linear ODE

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2nd Order Inhomogeneous Linear ODE

Example

$$y'' + 2y' + y = e^x$$
$$y'' - x^{-1}y' + x^{-2}y'' = e^x$$
$$y'' - \sin(x)y' + (x)y = x^2 - (x+1)$$

Uniqueness Revisited

Uniqueness

$$y'' + p(x)y' + q(x)y = r(x)$$
$$y(x_0) = K_0,\ y'(x_0) = K_1$$

Interval I:

1. $p(x)$, $q(x)$, $r(x)$ continuous on I
2. $x_0 \in I$

Solution exists and is unique

Miracle Proof of Uniqueness

y_1, y_2 both solutions

$$\Longrightarrow$$

$y_1 - y_2$ solution of homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0$$
$$y(x_0) = 0,\ y'(x_0) = 0$$

$$\Longrightarrow$$

$$y_1 = y_2$$

Miracle Proof of Uniqueness

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Suggests: Homogeneous may provide useful information about inhomogeneous.

Solving ODEs: Strategy One: Just Guess

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(Intelligently, of course)

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Basic Principle

Put in (almost) what you want out.

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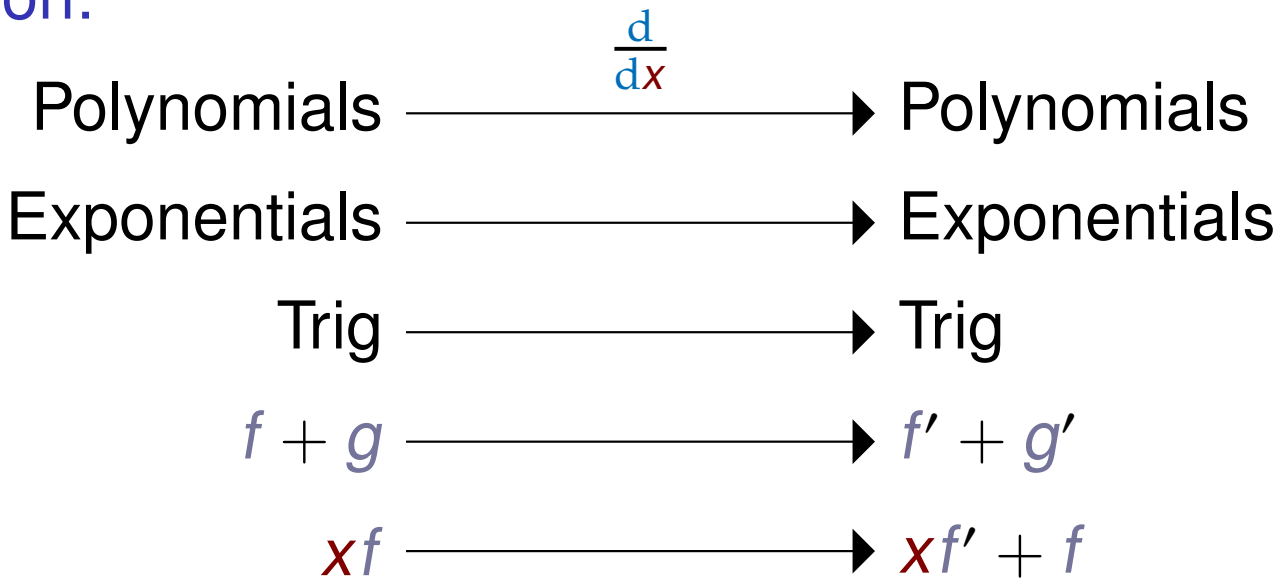
Based on:



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Try: $\frac{1}{4}e^x$.

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$$y'' + 2y' + y = e^x$$

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Try: $\frac{1}{4}e^x$.

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Hence: $\frac{1}{4}e^x$ is a solution.

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$$y'' + 2y' + y = e^x$$

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(What about other solutions? Come back to that later.)

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Try xe^x:

$$\text{LHS}: \quad xe^x + 2e^x - xe^x = 2e^x$$

Examples

$$y'' - y = e^x$$

Try e^x:

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- What next?
- Use trick from homogeneous case: multiply by x!

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Hence: $\frac{1}{2}xe^x$ is a solution.

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$$y'' - 2y' + y = e^x$$

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Examples

$$y'' + 2y' + y = x$$

Try x:

$$\text{LHS}: \quad 0 + 2 + x = 2 + x$$

Try $x - 2$:

$$\text{LHS}: \quad 0 + 2 + x - 2 = x$$

Solution: $x - 2$

Intelligent Guesswork

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  1. Low degree polynomials, a, x, x^2
  2. Solutions of homogeneous ODE

Summary

- Wronskian:

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

- Use Wronskian to test for linear independence
- Inhomogenous ODEs:

$$y'' + p(x)y' + q(x)y = r(x)$$

- Solve by intelligent guesswork