TMA4115 Matematikk 3

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Lecture 5: The Grand Tour

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Recap

- Studying ODEs of form:

$$y'' + p(x)y' + q(x)y = 0$$

- Reduction of order: use one solution to get another
- Constant coefficients: exponential solutions

Key Points

- Another type of ODE: Euler–Cauchy
- Idea of uniqueness
- Test for linear independence
- (Introducing: the Wronskian!)

ODEs and Polynomials

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consider

$$\lambda^2 + a\lambda + b$$

(auxilliary or characteristic equation)

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$$\text{ODEs} \longleftrightarrow \text{Polynomials}$$

Not a coincidence (learn more in *Fourier analysis*).

Euler–Cauchy ODEs

Introducing . . .

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- Why?

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- Why? Because we can solve it.
- Anticipate “issues” at 0!
- Similar to $y'' + ay' + by = 0$

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- Degree shift: try x^m

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- Auxilliary equation:

$$m^2 + (a-1)m + b$$

Solution

Solution of Cauchy–Euler

If m is a root of

$$m^2 + (a - 1)m + b$$

then x^m satisfies

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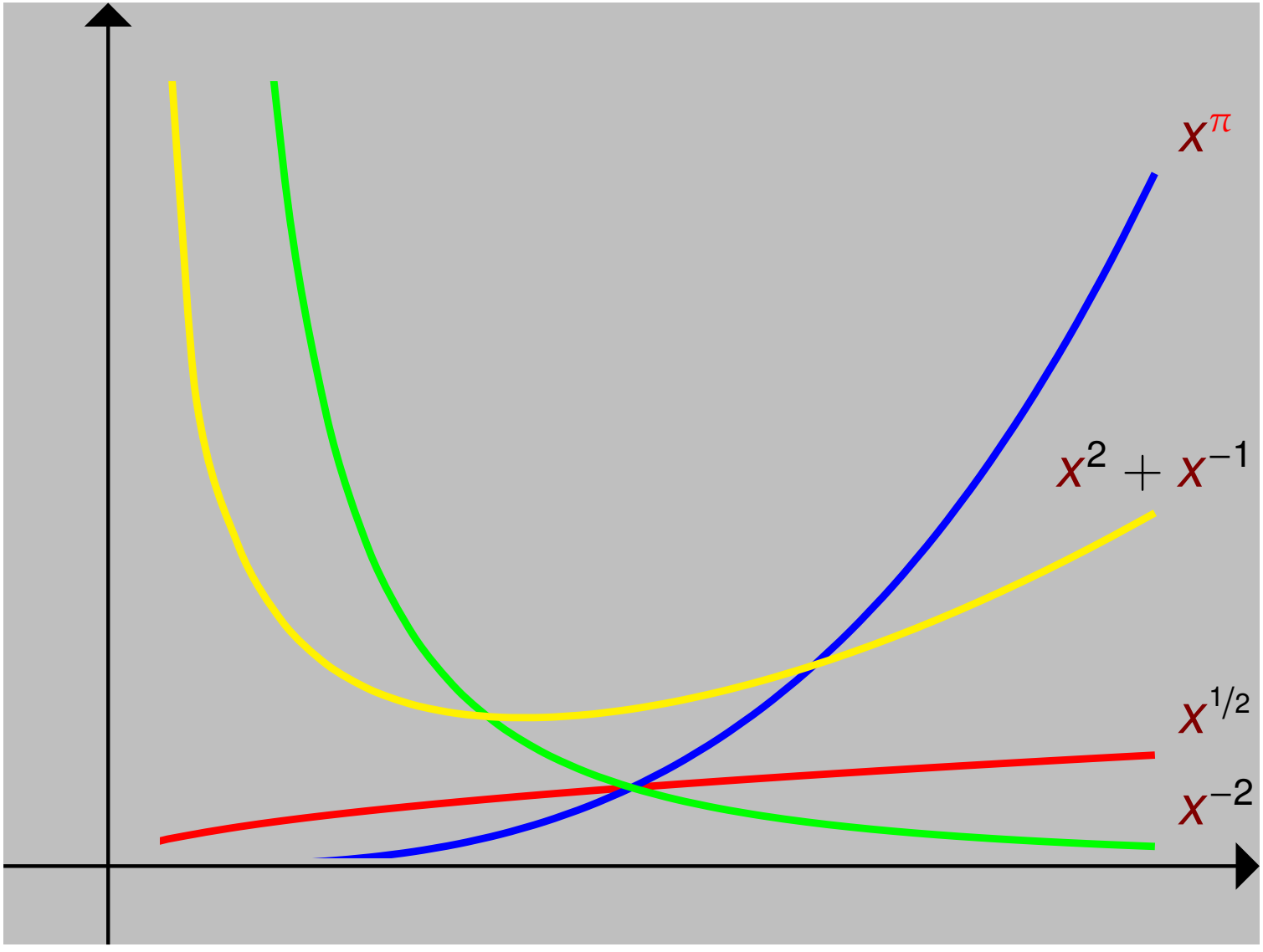
- Note: $m \in \mathbb{R}$. *Not* $\mathbb{Z}$*!*
- 2 distinct real roots $\Longrightarrow$ 2 distinct solutions
- Potential problems at $x = 0$ (as expected)
- 1 real root or 2 complex roots not much harder
  But not on syllabus. . .

Pretty Pictures



Spot the Difference

Compare and Contrast

$$x^2y'' + axy' + by = 0$$

$$y'' + ax^{-1}y' + bx^{-2}y = 0$$

Case Study

$$x^2 y'' - 3xy' + 3y = 0$$

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$$cx + dx^3$$

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So far so good

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Conclusion: No solutions!

Caveat Solvator

Important

Functions make sense at $x = 0$

$$\not\Longleftrightarrow$$

Solutions make sense at $x = 0$

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Solution

- No substantial difference

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Solution

- No substantial difference
- Standard form clearer hence better

What Is an ODE?

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A way of specifying a curve by its derivatives.

Key word: specifying.
If it doesn't specify, it isn't (so) useful!

Uniqueness or Existence

Theorem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(x_0) = K_0, y'(x_0) = K_1$$

Interval I:

- $p(x)$, $q(x)$ *continuous on* I
- $x_0 \in I$.

Then IVP has a solution and it is unique.

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Then IVP has a solution and it is unique.

Existence There is a solution

Uniqueness The solution is unique

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Then IVP has a solution and it is unique.

Existence There is at least one solution

Uniqueness There is at most one solution

Uniqueness By Squeezing

Key Property of Continuous Functions

$f : [a, b] \to \mathbb{R}$ continuous

$\exists t_0 \in [a, b] : |f(t_0)| \geq |f(t)| \forall t \in [a, b]$

i.e. $|f|$ achieves its maximum

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Apply to: $p(x)$, $q(x)$, $y'(x)$.

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Simplifying Assumptions:

1. Work on $[0, 1]$

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2. $|p(x)|, |q(x)| \leq 1/4$ all $x \in [0, 1]$

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Simplifying Assumptions:

1. Work on $[0, 1]$
2. $|p(x)|, |q(x)| \leq 1/4$ all $x \in [0, 1]$
3. Consider IVP with $y(0) = 0$, $y'(0) = 0$

Know that $y(x) = 0$ is a solution.

Goal: Show it is the only solution.

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$$y'' + p(x)y' + q(x)y = 0, \quad y(0) = 0,\ y'(0) = 0$$

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By Key Property of Continuous Functions:
there is $x_0 \in [0, 1] : |y'(x_0)| \geq |y'(x)|$ all $0 \leq x \leq 1$

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Integration $\Longrightarrow |y(x)| \leq |y'(x_0)|$ also

Uniqueness By Squeezing

Then:

$$y'(x_0) = \int_0^{x_0} y''(x)\mathrm{d}x$$

Uniqueness By Squeezing

Then:

$$y'(x_0) = \int_0^{x_0} y''(x)\mathrm{d}x$$
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But $|y'(x)| \leq |y'(x_0)|$ so $y'(x) = 0$ for all $x \in [0, 1]$!
So $y(x) = 0$ and solution is unique.

Uniqueness Facts

$$y'' + p(x)y' + q(x)y = 0$$

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2. $y_1 \neq 0$, y_2 solutions
   $y_1(x_0) = 0 = y_2(x_0)$ some x_0
   $\Longrightarrow y_2 = k y_1$ some $k \in \mathbb{R}$

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   Non-zero $\Longrightarrow y_1'(x_0) \neq 0$
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3. Same with $y_1'(x_0) = 0 = y_2'(x_0)$.

Uniqueness Example

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Solutions: $\sin(2x)$, $\cos(2x)$, $\sin(x)\cos(x)$.

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Uniqueness Example

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Solutions: $\sin(2x)$, $\cos(2x)$, $\sin(x)\cos(x)$.

1. $\sin(2x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\sin(2x)$ never simultaneously zero
2. $\sin(2\pi) = 0 = \sin(\pi)\cos(\pi) \Longrightarrow$
   $\sin(2x) = 2\sin(x)\cos(x)$

Linear Independence

Uniqueness motivates linear independence:

Theorem

y_1, y_2 *linearly independent solutions*

$\Longleftrightarrow$

All solutions uniquely of form

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Recall:

Definition

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Recall:

Definition

y_1, y_2 *linearly independent*

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Problem: Have to test all a, b and all x

Testing Linear Independence

y_1, y_2 not linearly independent

$\Longleftrightarrow$

one is a multiple of the other

$\Longleftrightarrow$

$y_2 = k y_1$ or $y_1 = 0 \Longleftrightarrow$

$y_2(x) = k y_1(x) \forall x$ or $y_1(x) = 0 \forall x$

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Problem: k unknown *if it exists*

Solution: use a x_0 to limit choices

Testing Linear Independence

$$\Big(y_1(x_0) = 0 \text{ and } y_1'(x_0) = 0\Big) \Longrightarrow \Big(y_1 = 0\Big) \Longrightarrow \text{dependent.}$$

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$\Big(y_1(x_0) = 0 \text{ and } y_1'(x_0) = 0\Big) \Longrightarrow \Big(y_1 = 0\Big) \Longrightarrow$ dependent.
So if $y_1 \neq 0$, at least one of

$$\frac{y_2(x_0)}{y_1(x_0)}, \qquad \frac{y_2'(x_0)}{y_1'(x_0)}$$

is well-defined and if $y_2 = k y_1$ then k is one of them.

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Tests

1. Is $y_1 = 0$?

Testing Linear Independence

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is well-defined and if $y_2 = k y_1$ then k is one of them.

Tests

1. Is $y_1 = 0$?
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$\Big(y_1(x_0) = 0 \text{ and } y_1'(x_0) = 0\Big) \Longrightarrow \Big(y_1 = 0\Big) \Longrightarrow$ dependent.
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Yes to any $\Longrightarrow$ dependent No to all $\Longrightarrow$ independent

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Testing Linear Independence

Claim

y_1 and y_2 linearly independent

$$\Longleftrightarrow$$

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) = 0$$

for some x_0

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How can it be zero?

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1. $y_1(x_0) = 0$ and $y_1'(x_0) = 0 \Longrightarrow$ test 1 holds

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How can it be zero?

1. $y_1(x_0) = 0$ and $y_1'(x_0) = 0 \Longrightarrow$ test 1 holds
2. $y_1(x_0) \neq 0 \Longrightarrow$ test 2 holds
3. $y_1(x_0) \neq 0 \Longrightarrow$ test 3 holds

Introducing . . . The Wronskian

Definition

The Wronskian of two functions

y_1, y_2

is

$$W(y_1, y_2) := y_1 y_2' - y_1' y_2$$

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3. $W(\cos(x), \sin(x)) =$
   $\cos(x)\cos(x) - (-\sin(x)\sin(x)) = 1$

Wronskian and Linear Independence

$$y'' + p(x)y' + q(x)y = 0$$

y_1 and y_2 solutions of ODE.
y_1, y_2 linearly independent

$$\Longleftrightarrow$$

$W(y_1, y_2)(x) = 0$ for all x

$$\Longleftrightarrow$$

$W(y_1, y_2)(x_0) = 0$ for some x_0

The Norwegian Connection

Really Weird Fact

$W(y_1, y_2)$ depends on ODE and almost not on y_1 and y_2!

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Solutions: $ax + bx^2$

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$$\begin{aligned} W(a_1x + b_1x^2, a_2x + b_2x^2) &= (a_1x + b_1x^2)(a_2 + 2b_2x) \\ &\quad - (a_1 + 2b_1x)(a_2x + b_2x^2) \\ &= a_1a_2x + (2a_1b_2 + a_2b_1)x^2 + 2b_1b_2x^3 \\ &\quad - a_1a_2x - (a_1b_2 + 2a_2b_1)x^2 - 2b_1b_2x^3 \\ &= (a_1b_2 - a_1b_1)x^2 \end{aligned}$$

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Always a multiple of x^2!

Abel's Identity

$$
\begin{aligned}
W(y_1, y_2)' &= y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' \\
&= y_1(-p y_2' - q y_2) - (-p y_1' - q y_1) y_2 \\
&= -p(y_1 y_2' - y_1' y_2) \\
&= -p W(y_1, y_2)
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1st Order Linear ODE!

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1st Order Linear ODE!
Solution:

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Curiousity for now, useful later.

Summary

- Euler–Cauchy ODEs provide useful examples of solvable ODEs
- Uniqueness is extremely useful
- Standard form important to see extent of ODE
- Linear independence testable by computing

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)$$

  at any x_0 in the interval of interest
- Above formula will be useful, so give it a name: Wronskian