TMA4115 Matematikk 3

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Lecture 4: Put a Spring in Your ODE

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Key Points

- Reduction of Order: An Example
- Solving Linear ODEs with Constant Coefficients

Detailed Recap

- 2nd Order Linear Homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

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- Solution:

Detailed Recap

- 2nd Order Linear Homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

- Initial conditions:

$$y(0) = a, \qquad y'(0) = b \tag{2}$$

- Problem: Solve it
- Solution:
  1. Find a function y such that 1 and 2 hold.

Detailed Recap

- 2nd Order Linear Homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

- Initial conditions:

$$y(0) = a, \qquad y'(0) = b \tag{2}$$

- Problem: Solve it
- Solution:
  1. Find 2 functions, y_1, y_2, such that 1 holds for both.

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- Initial conditions:

$$y(0) = a, \qquad y'(0) = b \tag{2}$$

- Problem: Solve it
- Solution:
  1. Find 2 functions, y_1, y_2, such that 1 holds for both.
  2. Show linear indepdenence

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- Problem: Solve it
- Solution:
  1. Find 2 functions, y_1, y_2, such that 1 holds for both.
  2. Show linear indepdenence
  3. Solve

$$cy_1(0) + dy_2(0) = a$$
$$cy_1'(0) + dy_2'(0) = b$$

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$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

- Initial conditions:

$$y(0) = a, \qquad y'(0) = b \tag{2}$$

- Problem: Solve it
- Solution:
  1. Find 1 function, y_1, such that 1 holds.
  2. Substitute $y = uy_1$ and solve for u'
  3. Solve

$$cy_1(0) + dy_2(0) = a$$
$$cy_1'(0) + dy_2'(0) = b$$

Reduction of Order: Example

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- Not in standard form.

Reduction of Order: Example

$$y'' - \frac{x^2 - 2}{x^2 - 2x} y' + \frac{2(x-1)}{x^2 - 2x} y = 0, \quad y(1) = 1,\ y'(1) = 0.$$

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  Not a problem!
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  (Avoid 0 here as ODE does not make sense there.)

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- Start by putting in simple functions.

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$$0 = k(k-1)x^{k-2} - k\frac{x^2 - 2}{x^2 - 2x} x^{k-1} + \frac{2(x-1)}{x^2 - 2x} x^k$$

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$$\Rightarrow 0 = (2-k)x^{k+1} + (k(k-1)-2)x^k + 2k(k-2)x^{k-1}$$

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- So x^2 works.

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$$\begin{aligned} 0 &= y'' - \frac{x^2 - 2}{x^2 - 2x} y' + \frac{2(x-1)}{x^2 - 2x} y \\ &= u''x^2 + 4u'x + 2u \\ &\quad - \frac{x^2 - 2}{x^2 - 2x}(u'x^2 + 2ux) + \frac{2(x-1)}{x^2 - 2x} ux^2 \end{aligned}$$

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Fast Track: Only terms in u'' and u' will "survive".

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$$0 = u''x^2 + \left(4x - \frac{x^2(x^2-2)}{x^2 - 2x}\right)u'$$

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$$0 = u'' + \left(4x^{-1} - \frac{x^2 - 2}{x^2 - 2x}\right)u'$$

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Note obvious solution: $u = \text{const}$.
Gives x^2 as solution of original ODE — which we already knew! So ignore.

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$$y = x^2 u = \mathrm{e}^x$$

Reduction of Order: Example

$$(x^2 - 2x)y'' - (x^2 - 2)y' + 2(x - 1)y = 0$$

has solutions x^2, e^x

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has solutions x^2, e^x
(Ought to check . . . exercise!)
Initial conditions: $y(1) = 1$, $y'(1) = 0$
Substituting in:

$$y(x) = ax^2 + be^x$$
$$y'(x) = 2ax + be^x$$
$$1 = a + be$$
$$0 = 2a + be$$

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Obtain: $y(x) = -x^2 + 2e^{x+1}$

Reduction of Order: Conclusion

- Starts with a guess, y_1
- After that, algorithmic (though can be complicated!)
- Key step: substitute $y = uy_1$ and solve for u'

Simple Harmonic Motion

Introducing . . .

A very special ODE:

$$y'' + ay' + by = 0$$

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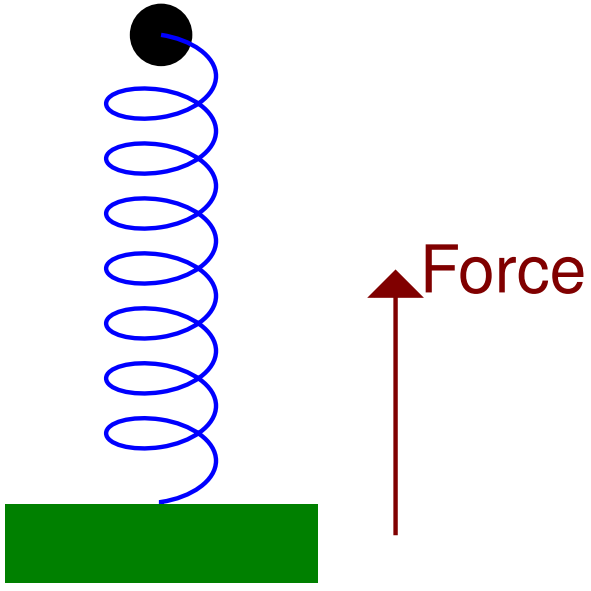
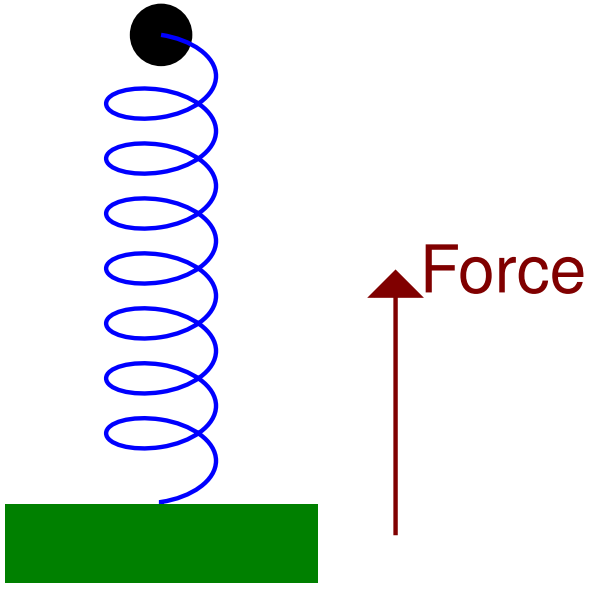
Occurs in models of many physical systems

Spring Dynamics

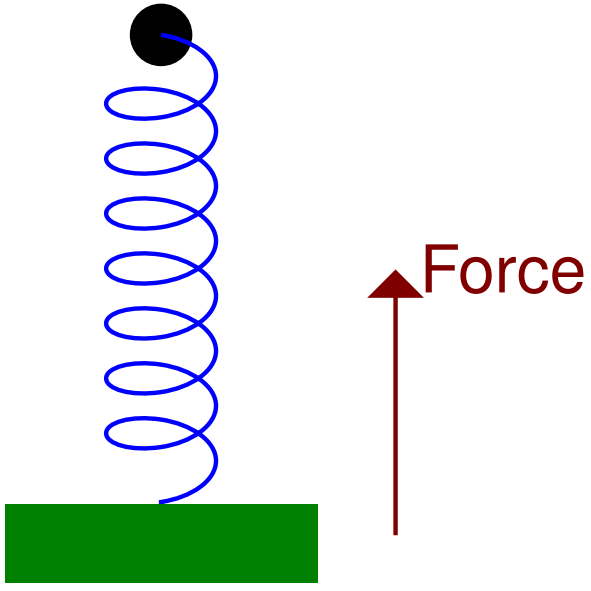


Spring Dynamics



- Newton: $F_{\text{total}} = mx''$

Spring Dynamics



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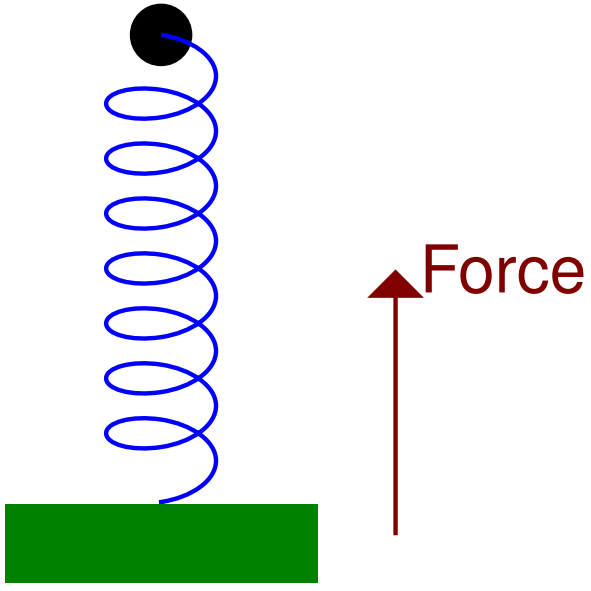
- Hooke: $F_{\text{spring}} = -x$

Spring Dynamics



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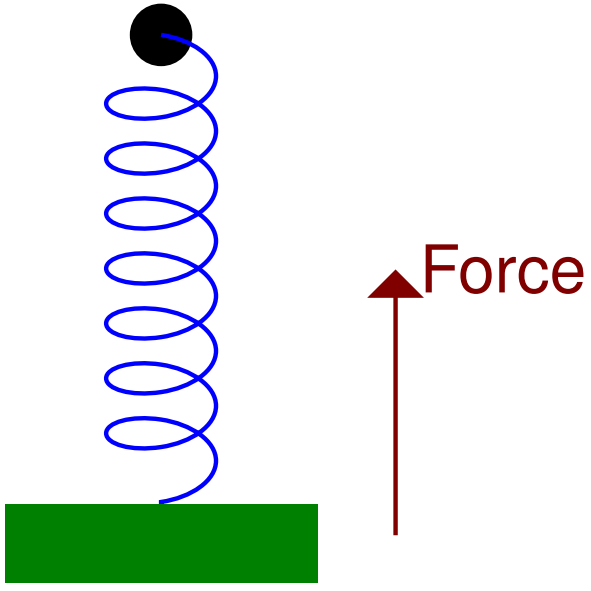
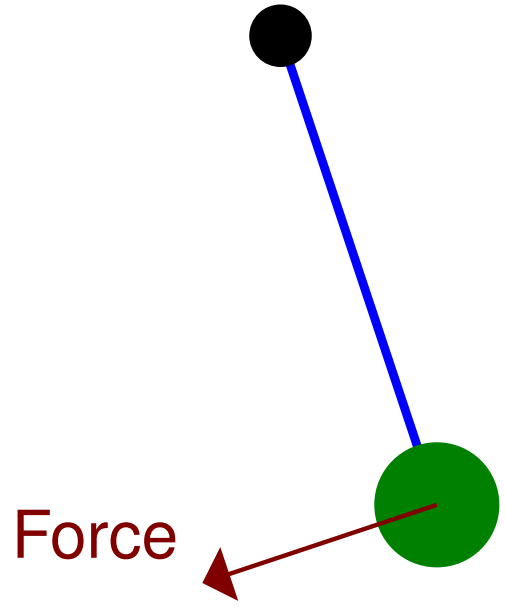


- Newton: $F_{\text{total}} = mx''$
- Hooke: $F_{\text{spring}} = -x$
- Friction: $F_{\text{friction}} = -cx'$
- In total:

$$mx'' + cx' + kx = 0$$

Pendular Dynamics



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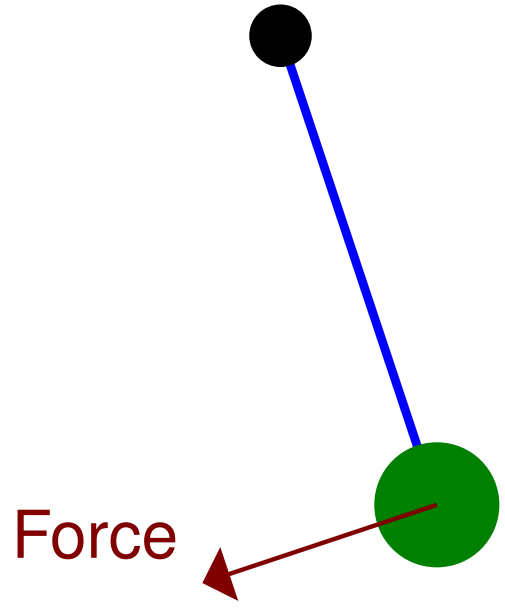

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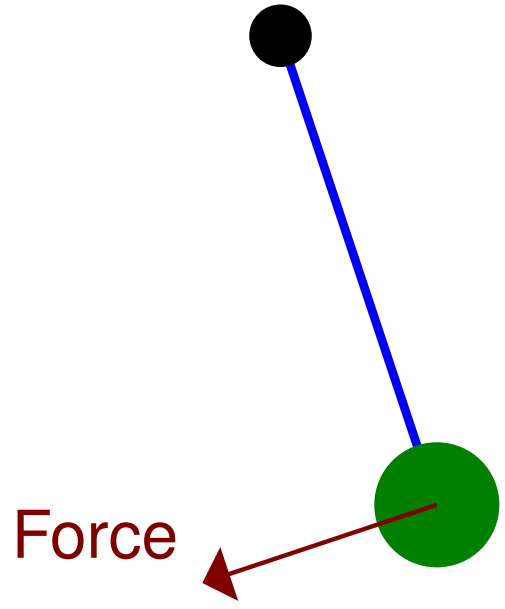

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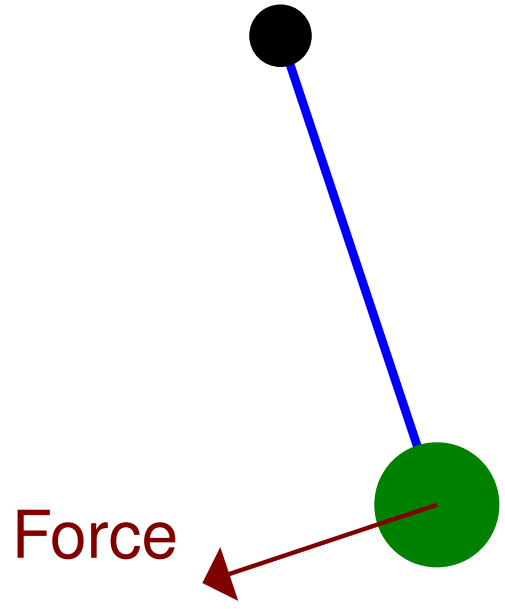

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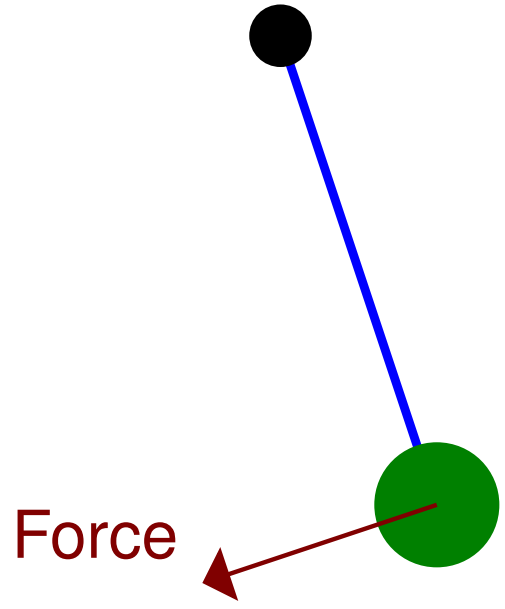
Pendular Dynamics



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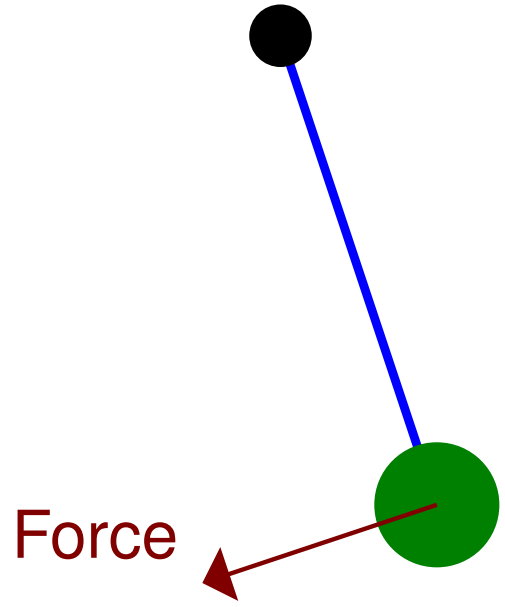

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- In total:

$$l\theta'' + c\theta' + g\theta = 0$$

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True if and only if

$$\lambda^2 + a\lambda + b = 0$$

Solutions: Grand Claim

$$y'' + ay' + by = 0$$

has solutions

$$e^{\lambda_1 x}, e^{\lambda_2 x}$$

where λ_1, λ_2 roots of

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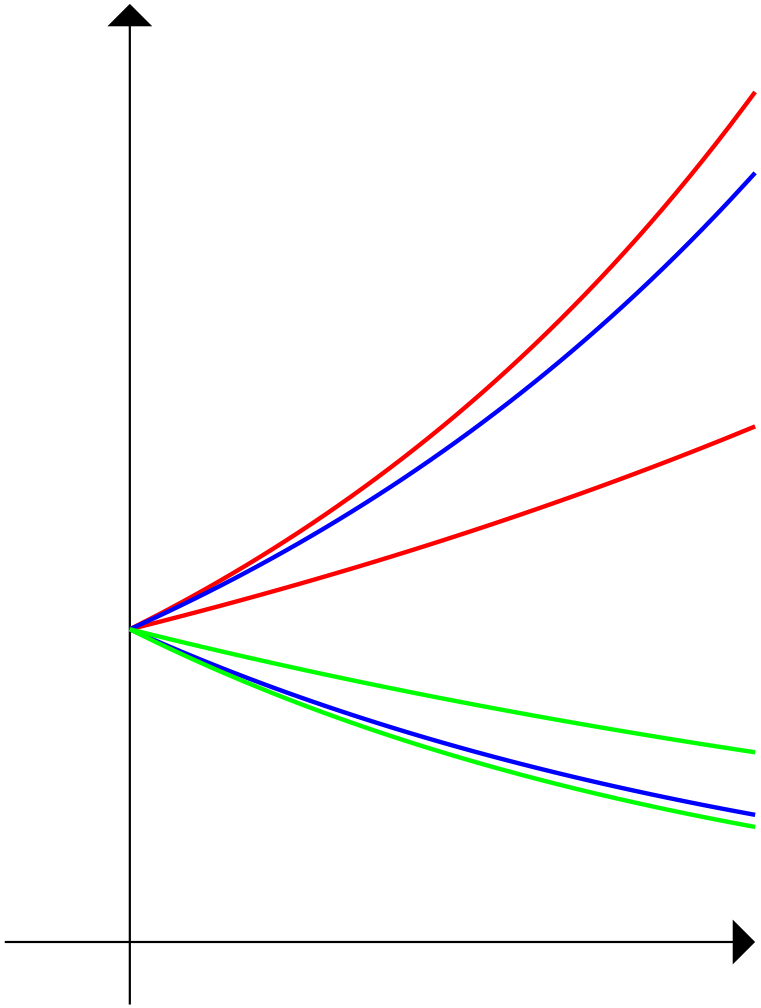

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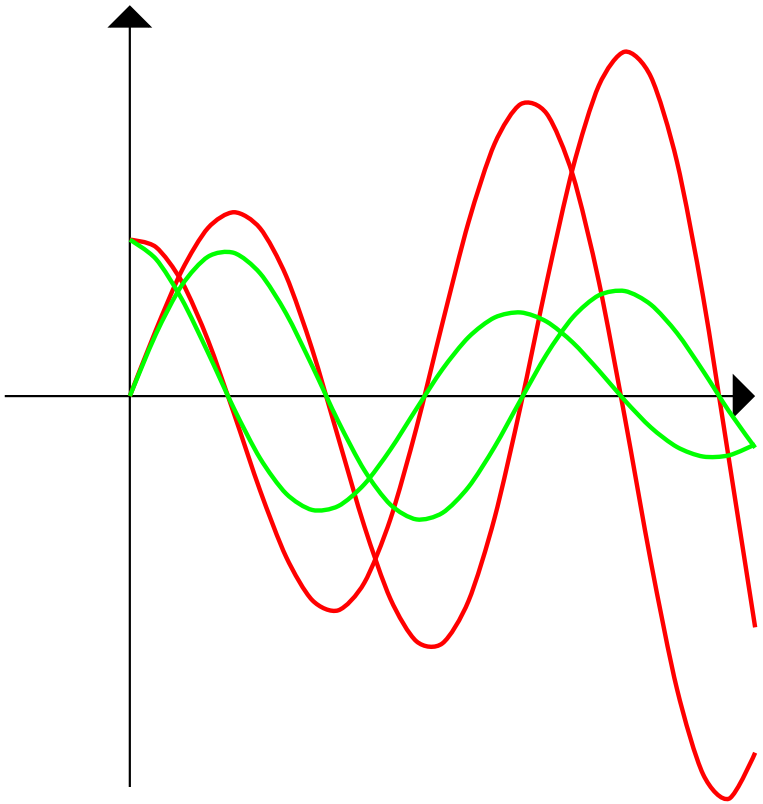

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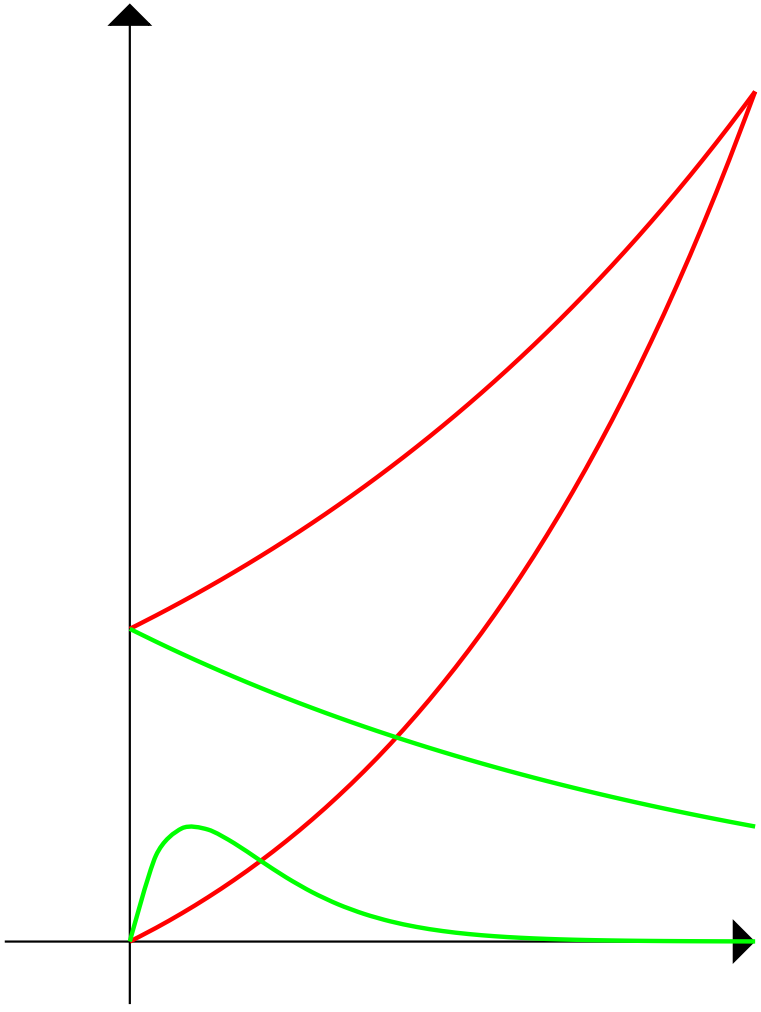
Solution: $u = x$, hence $xe^{\lambda x}$ is the other solution.

Repeated Roots



Summary

- Solving ODEs means Finding Solutions by any means
- Can use knowledge about ODE to help solve it
- Constant Coefficients has solutions

$$e^{\lambda_1 x}, \quad e^{\lambda_2 x}$$

$$e^{\lambda x}\cos(\omega x + \theta_0), \quad e^{\lambda x}\cos(-\omega x + \theta_0)$$

$$e^{\lambda x}, \quad x e^{\lambda x}$$