TMA4115 Matematikk 3

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Lecture 3: Two ODEs Are Better Than One

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Key Ideas

- ODEs occur everywhere
- Solving ODEs is an art
- “Ends justify the means”

Stories

A Mathematical Topic Has. . .

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1. Story

Stories

A Mathematical Topic Has. . .

1. Story
2. Techniques

Stories

A Mathematical Topic Has. . .

1. Story
2. Techniques
3. Examples

Differential Equations: The Story

The Good News

nice Differential Equation

$+$

precise initial conditions

$=$

unique solution.

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- "Nice": everything twice differentiable is enough
- "Precise": n bits of information for nth order

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- "Precise": n bits of information for nth order

The Bad News

No method to find solution!

Solving ODEs: An Analogy

Solve:

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Solving ODES $\longleftrightarrow$ Integration

Differentation is an algorithm
Integration is an art

Solving ODEs

The Problem

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million to 1

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Chance of needing to solve ODE with exact solution:

9 out of 10

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“Exact solution” really means “well-studied solution”!

Finding Solutions

Epilogue

However you find a solution

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- Technique = informed guesswork.

Types of ODEs

No single technique $\Longrightarrow$ sort ODEs by technique

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Introducing . . .

2nd Order Linear Ordinary Differential Equations

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2nd Order Linear Ordinary Differential Equations

2nd Order

Has y'' but not y'''

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Linear

Has y but not y^2.

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Example

$$y'' + \sin(x)y' + \cos(x)y = \tan(x)$$
$$y''y' + y^2 = \sin(x)$$

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(Not independent statements)

Decomposable

$$a, b \in \mathbb{R},$$
$$y'' + \sin(x)y' + \cos(x)y = 0$$
has unique solution $y_{a,b}(x)$ with
$$y_{a,b}(0) = a,\ y'_{a,b}(0) = b$$

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Non-Decomposable

Need to find $y_{a,b}$ for all $a, b \in \mathbb{R}$

Decomposable

Need to find $y_{a,b}$ for only two $a, b \in \mathbb{R}$

Homogeneous vs Inhomogeneous

Non-Linear

$$y''y' + y^2 = 0$$

No decomposition formula

Linear Inhomogeneous

$$y'' + \sin(x)y' + \cos(x)y = \tan(x)$$

Decomposition formula, but not simplest

Linear Homogeneous

$$y'' + \sin(x)y' + \cos(x)y = 0$$

Simple decomposition formula

The Problem

Given

$$y'' + p(x)y' + q(x)y = 0$$

find y_1, y_2 such that:

1. y_1, y_2 satisfy ODE
2. If y satisfies ODE then $y = ay_1 + by_2$, $a, b \in \mathbb{R}$
   say: y is a linear combination of y_1 and y_2

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Can replace 2 by:

2'. y_1 and y_2 are linearly independent:

$$ay_1 + by_2 = 0 \Longrightarrow a = b = 0$$

Neither is a constant multiple of the other.

An Actual Technique

Analogy

Solve:

$$x^2 + 4x - 12 = 0$$

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$$\frac{x^2 + 4x - 12 = 0}{x - 2} = x + 6$$

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So -6 is other solution.

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So -6 is other solution.
(Note: slightly dubious "divide by zero" step didn't matter!)

Reduction Of Order

Process

1. Guess a solution
2. Use to reduce to 1st order

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Note: 1st order $\Longrightarrow$ easy!

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Solve:

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$$y'' = u''\sin(x) + 2u'\cos(x) - u\sin(x)$$

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$$\begin{aligned} y' &= u'\sin(x) + u\cos(x) \\ y'' &= u''\sin(x) + 2u'\cos(x) - u\sin(x) \\ 0 = y'' + y &= u''\sin(x) + 2u'\cos(x) \end{aligned}$$

Example ctd

Obtain:

$$u''\sin(x) + 2u'\cos(x) = 0$$

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Still 2nd Order Linear ODE.

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$$U' + 2U\cot(x) = 0$$

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Solution:

$$U = e^{-2\int \cot(t)\,dt} = \frac{1}{\sin^2(x)}$$

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So:

$$u = \int U dt = \cot(x)$$

Hence:

$$y = \cot(x)\sin(x) = \cos(x)$$

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Key Idea

Use guessed solution to change:

2nd Order ODE $\longrightarrow$ Two 1st Order ODEs

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Key Idea

Use guessed solution to change:

2nd Order ODE $\longrightarrow$ Two 1st Order ODEs

Use y_1 to turn

$$y'' + p(x)y' + q(x) = 0$$

into

$$U' + \left(\frac{2y_1'}{y_1} + p\right)U = 0$$

$$u' = U$$

where $y_2 = uy_1$

A Final Interval

Solutions are often only defined on an interval.

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Intervals

$$\mathbb{R}$$
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$$[a, \infty)$$
$$(-\infty, a)$$
$$(-\infty, a]$$
$$(a, b)$$
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Interval often obvious, must contain “initial position”.

Summary

- 2nd order homogeneous linear ODEs: simple to solve
- 2 solutions to construct rest
- “Reduction of order”: use one solution to find another
- Finding solution is more important than method