TMA4115 Matematikk 3

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Trondheim

Spring 2010

Lecture 2: Mathematicians are $-e^{\pi i}$!

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15th January 2010

Recap

- Complex numbers $\mathbb{C}$
- $z = x + \mathrm{i}y$

  $x = \operatorname{Re} z$, $y = \operatorname{Im} z$
- Add, Subtract, Multiply, Divide

  just as in $\mathbb{R}$
- Complex conjugation: $\overline{x + \mathrm{i}y} = x - \mathrm{i}y$

Key Points

- Making sense of z^w
- Geometry of $\mathbb{C}$

The Power of Powers

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Why it might work

Multiplication is repeated addition and $\times$ extends
Powers are repeated multiplication

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Multiplication is repeated addition and $\times$ extends
Powers are repeated multiplication

Why it might not work

Taking powers is not symmetric
Powers in $\mathbb{R}$ already have problems

Problems with Powers

Non-Symmetry

$$a + b = b + a$$

$$a \times b = b \times a$$

$$a^b \neq b^a$$

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But sometimes it just works anyway!

Problems with Powers, pt II

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But not necessarily unique:

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So we proceed with caution!

Safe Territory

$$z^w$$

Definitely works for

- $w \in \mathbb{N}$: repeated multiplication
- $z \neq 0$, $w \in \mathbb{Z}$: reciprocal and repeated multiplication
- $z = -1$ and $w = \frac{1}{2}$

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Carpe Jugulum!

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Strategy

- Define e^w for as many $w \in \mathbb{C}$ as possible
- Define $\ln(z)$ for as many $z \in \mathbb{C}$ as possible
- Define $z^w := e^{w\ln(z)}$

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The Double Derivative Trick: If $f(t)$ satisfies $f' = \mathrm{i}f$ then it also satisfies

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Solutions:

$$f(t) = c\sin(t) + d\cos(t)$$

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- $f'(0) = \mathrm{i}f(0) = \mathrm{i} \Longrightarrow c = \mathrm{i}$

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$$f(t) = \cos(t) + \mathrm{i}\sin(t)$$

Conclusion

So define

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In particular:

$$\begin{aligned}
e^{i\pi} &= \cos(\pi) + i\sin(\pi) &&= -1 \\
e^{2i\pi} &= \cos(2\pi) + i\sin(2\pi) &&= 1 \\
e^{i\pi/2} &= \cos(\pi/2) + i\sin(\pi/2) &&= i \\
e^{i} &= \cos(1) + i\sin(1) &&= 0.54030 + i0.84147
\end{aligned}$$

Conclusion

So define

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Warning: use radians

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Logarithms!

After the ~~commercial~~ geometric break.

Geometry of $\mathbb{C}$

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Co-ordinates!

(Cartesian and polar)

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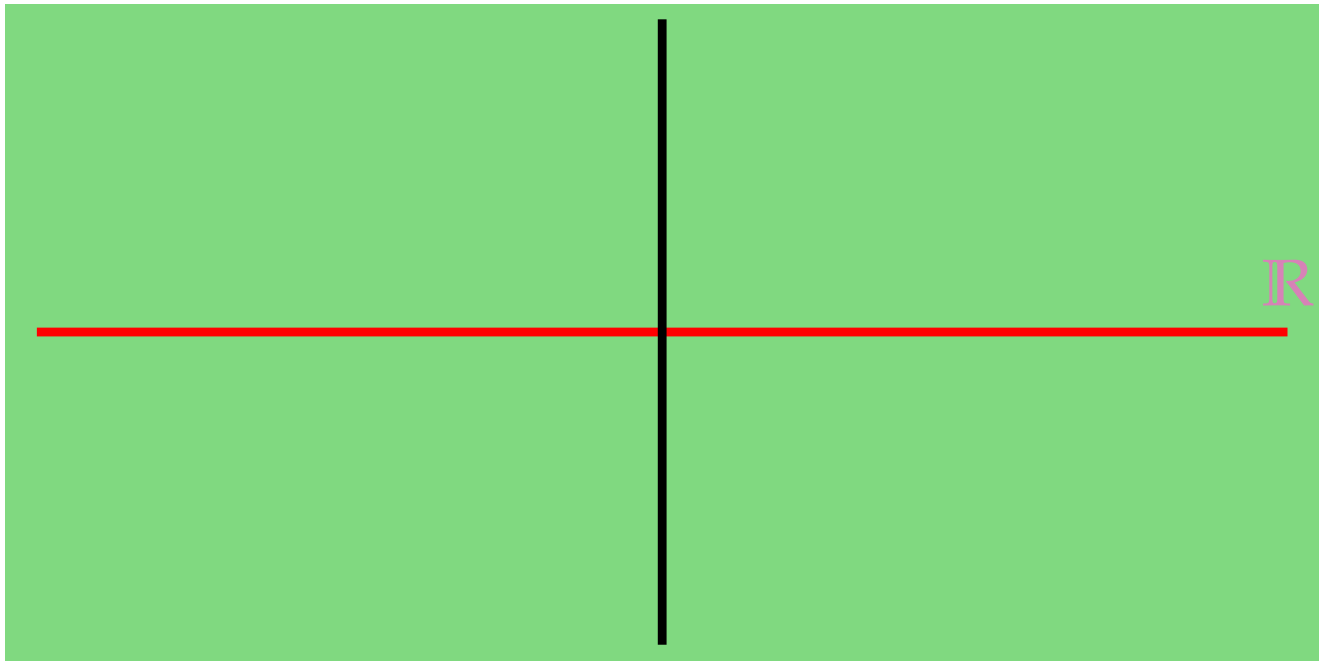

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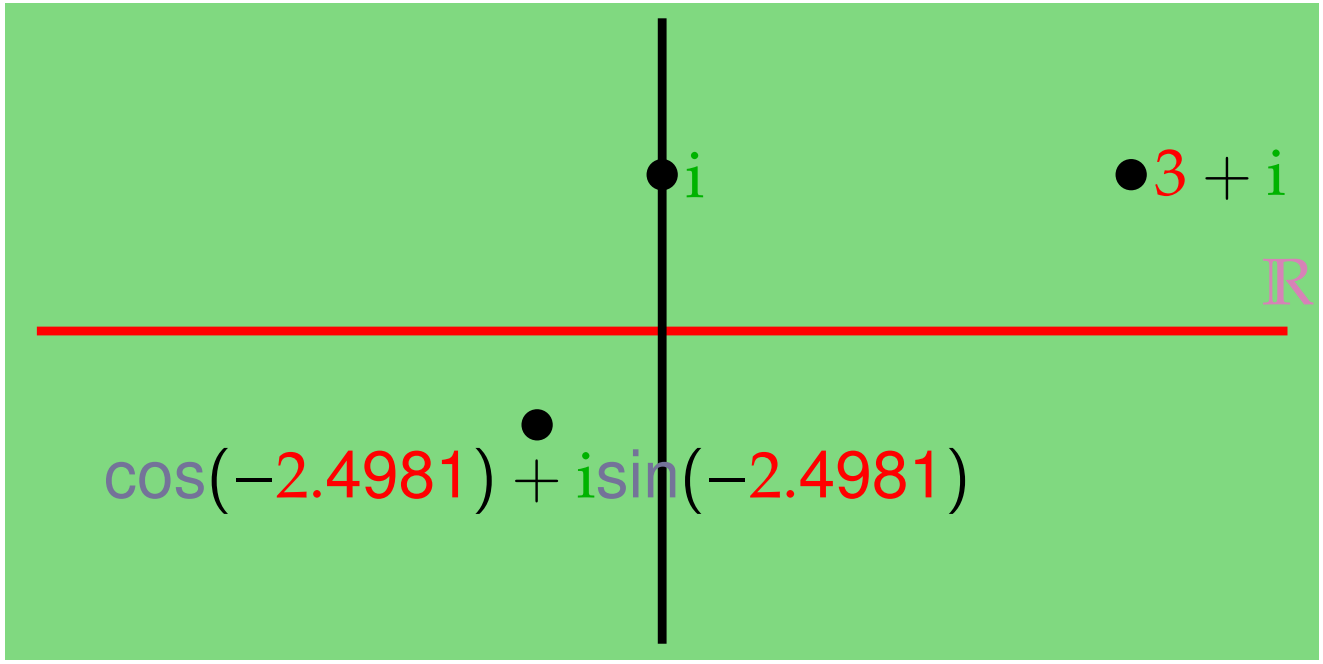

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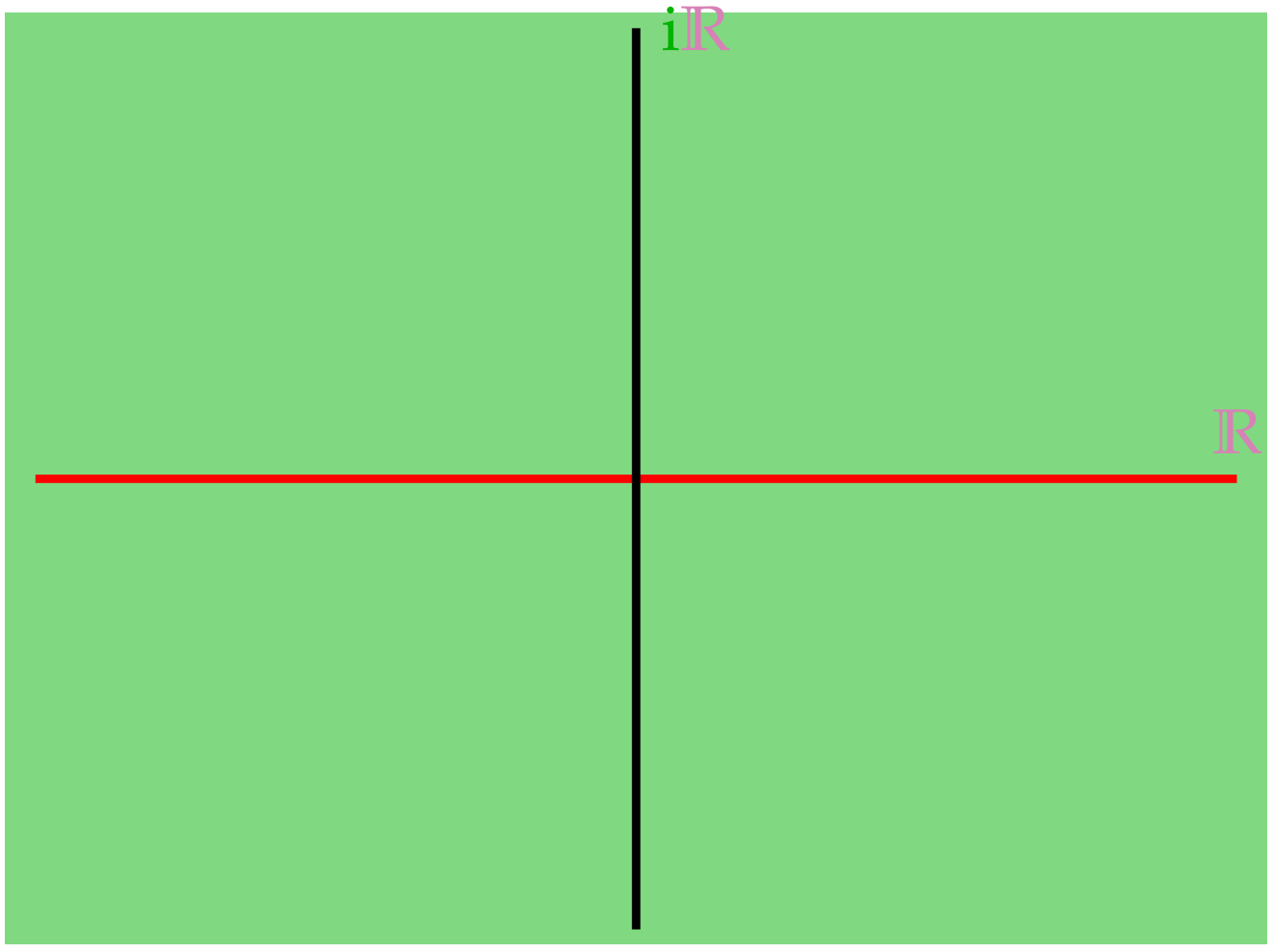

Addition, Subtraction

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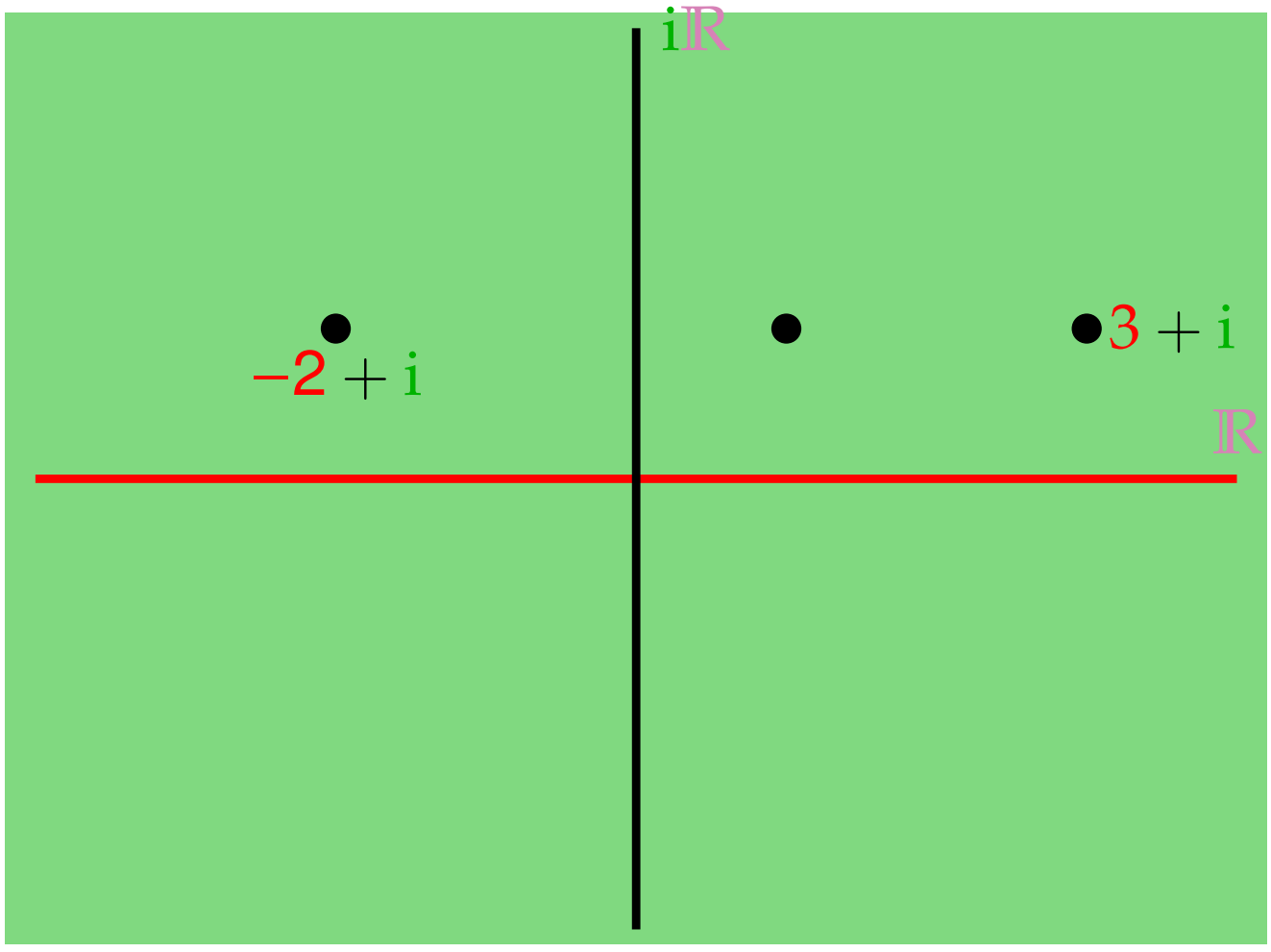

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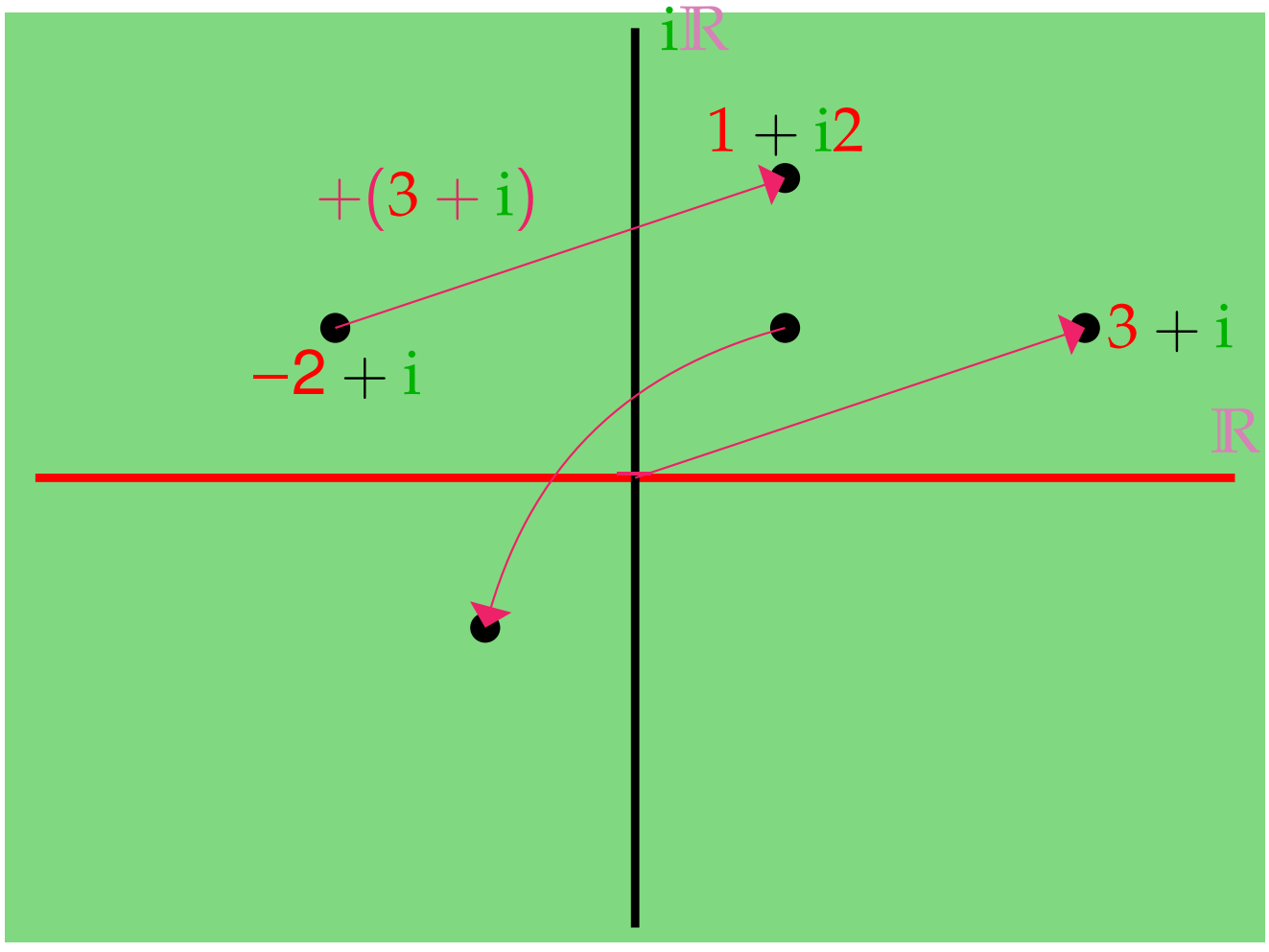

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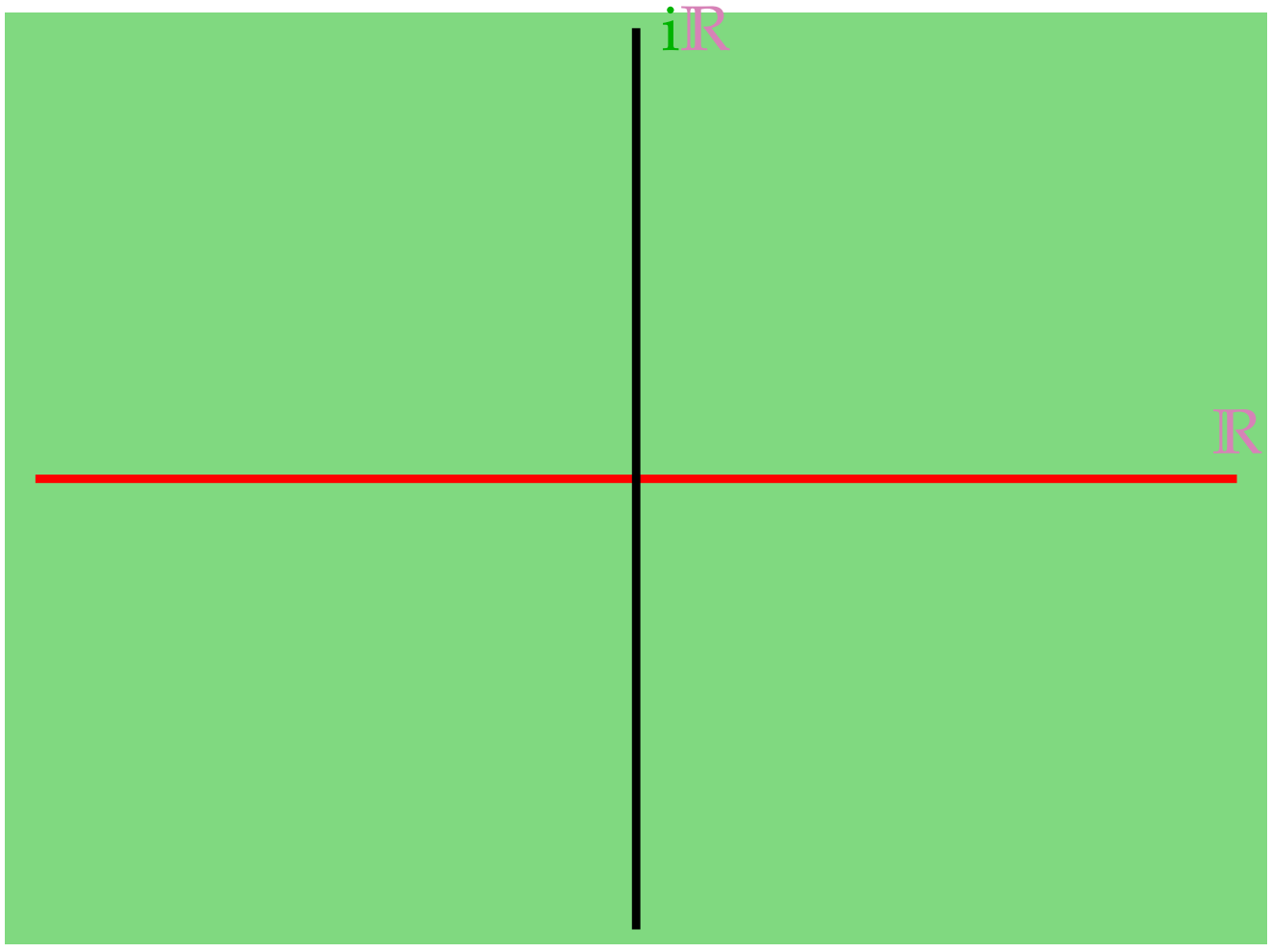

Complex conjugation, real and imaginary parts

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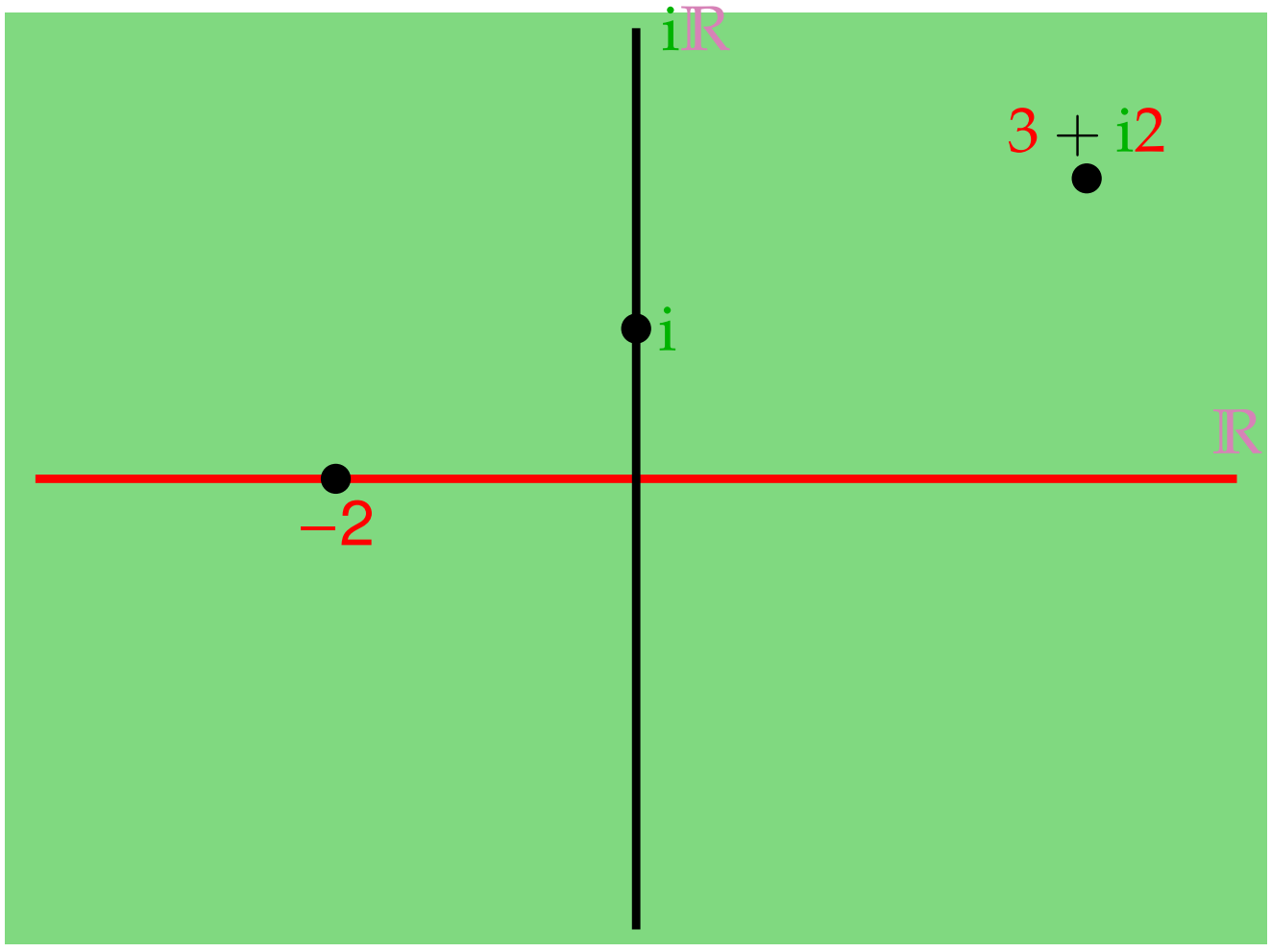

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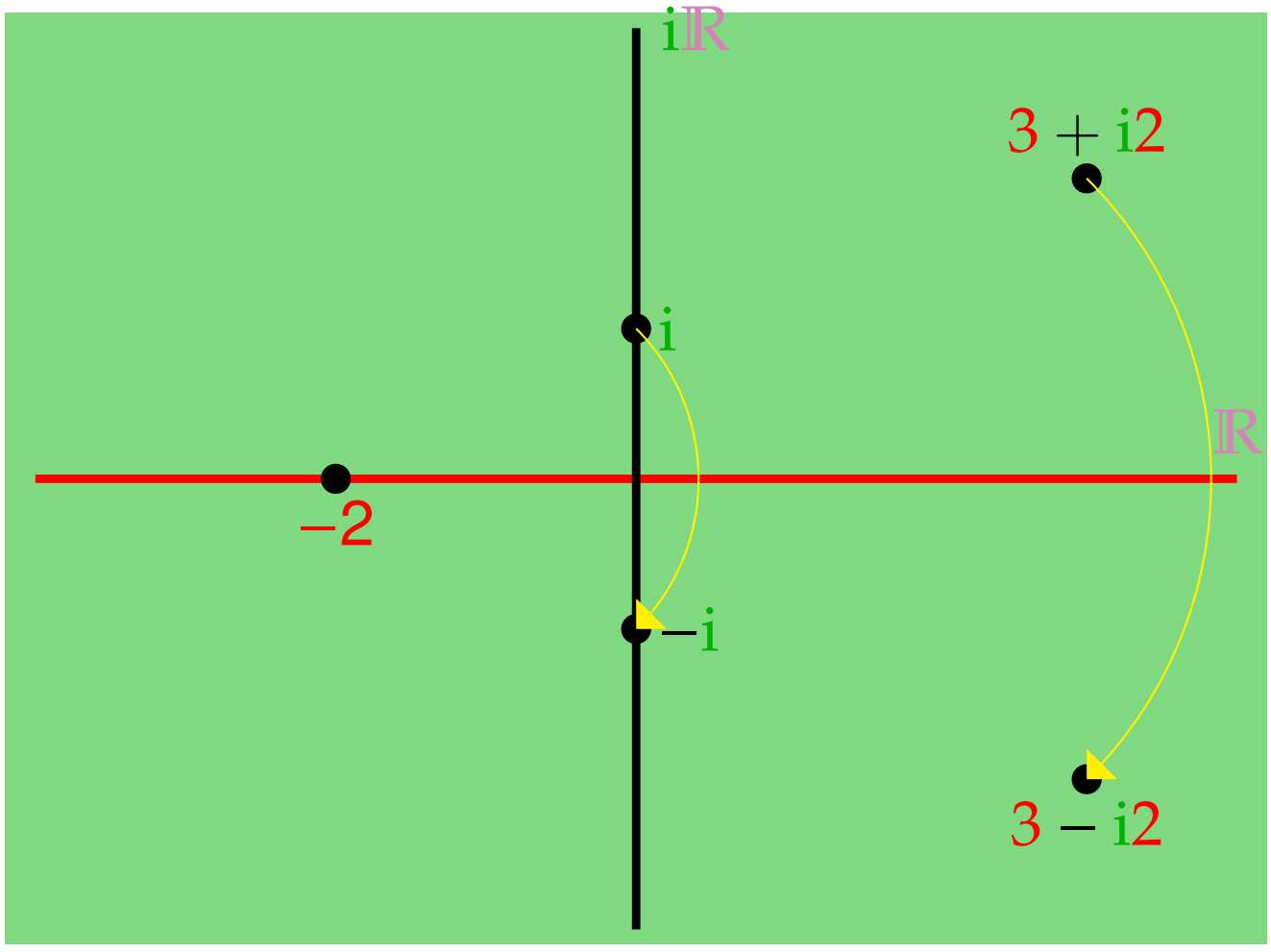

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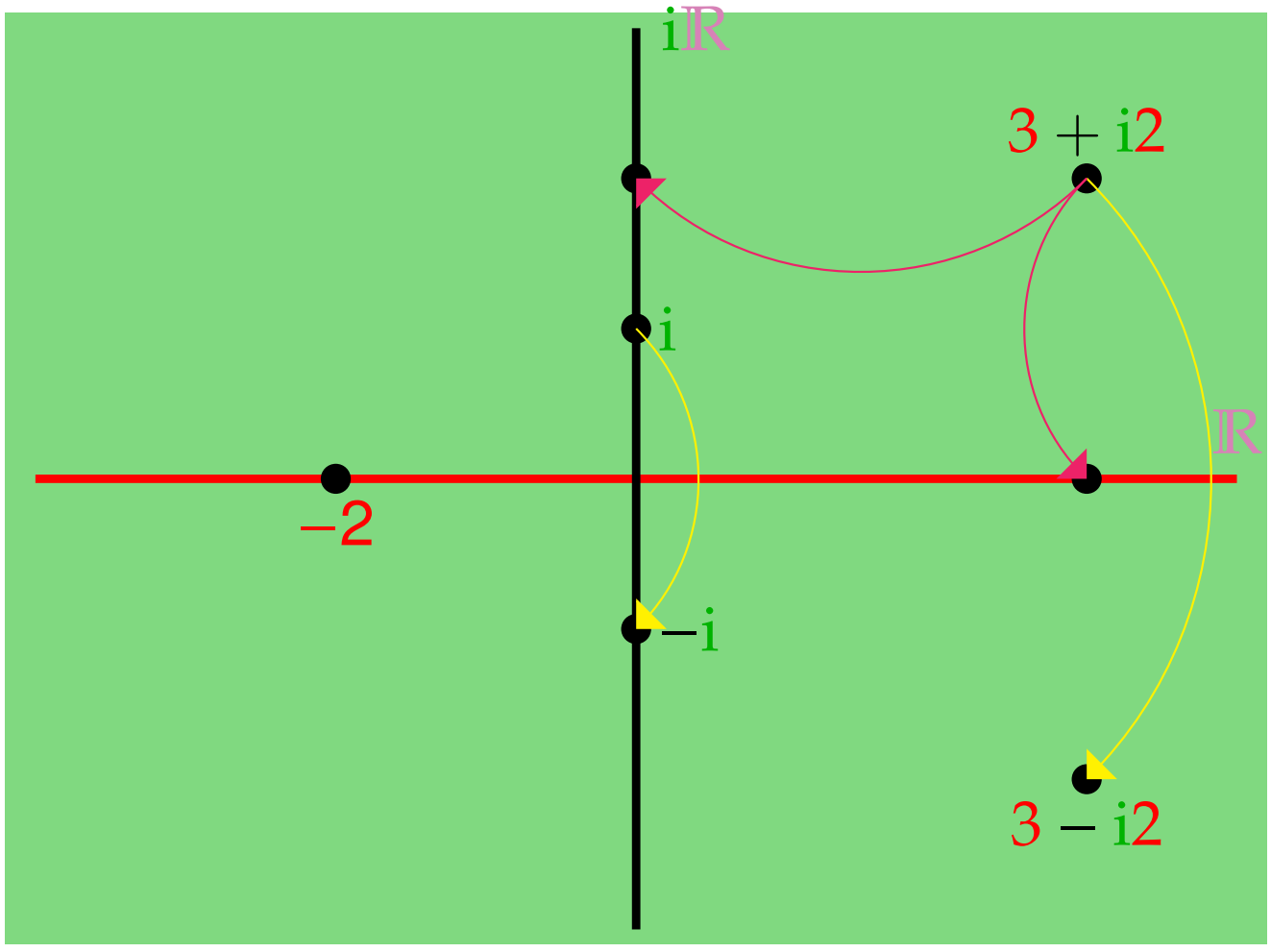

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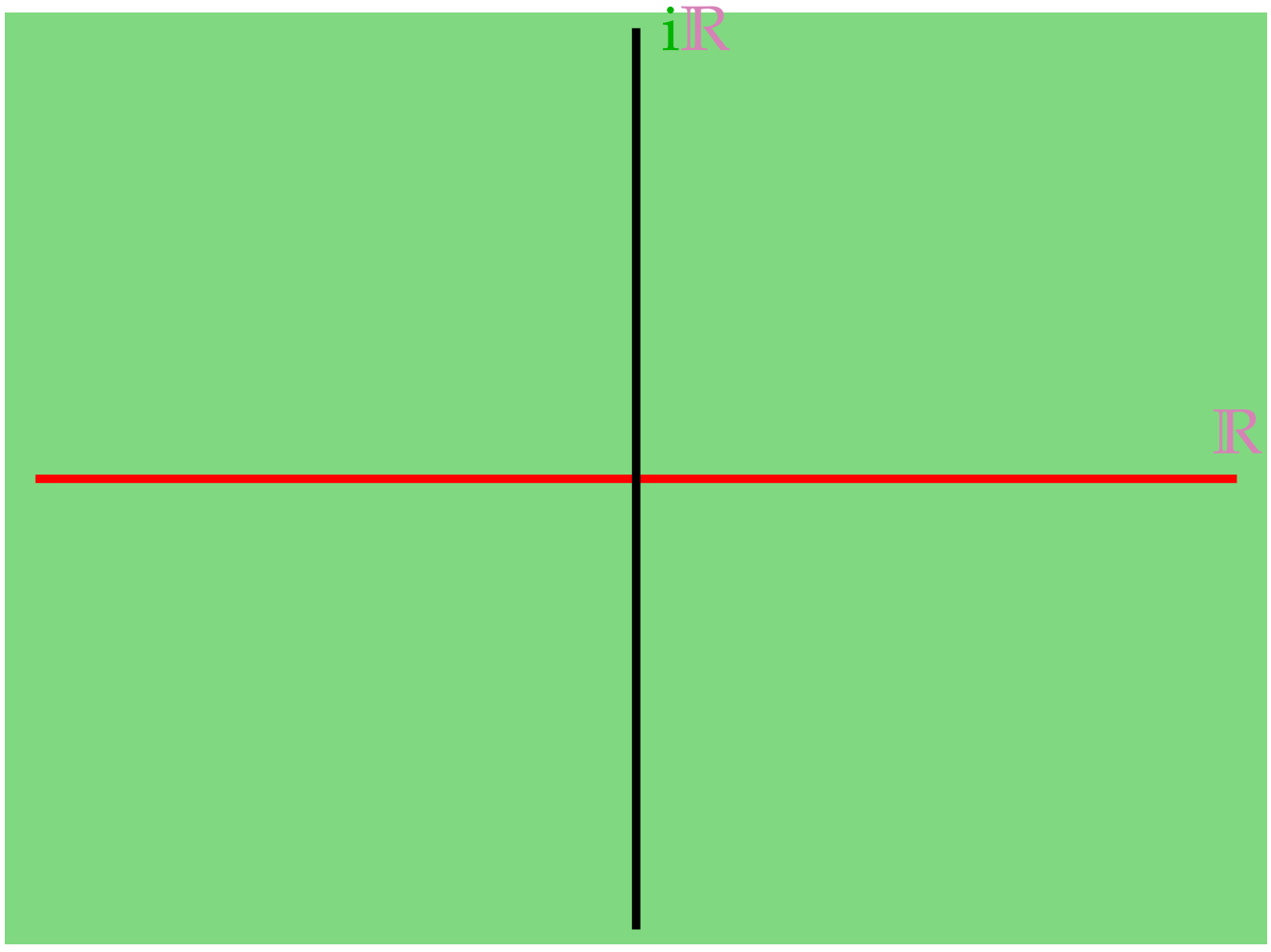

Polar Co-ordinates

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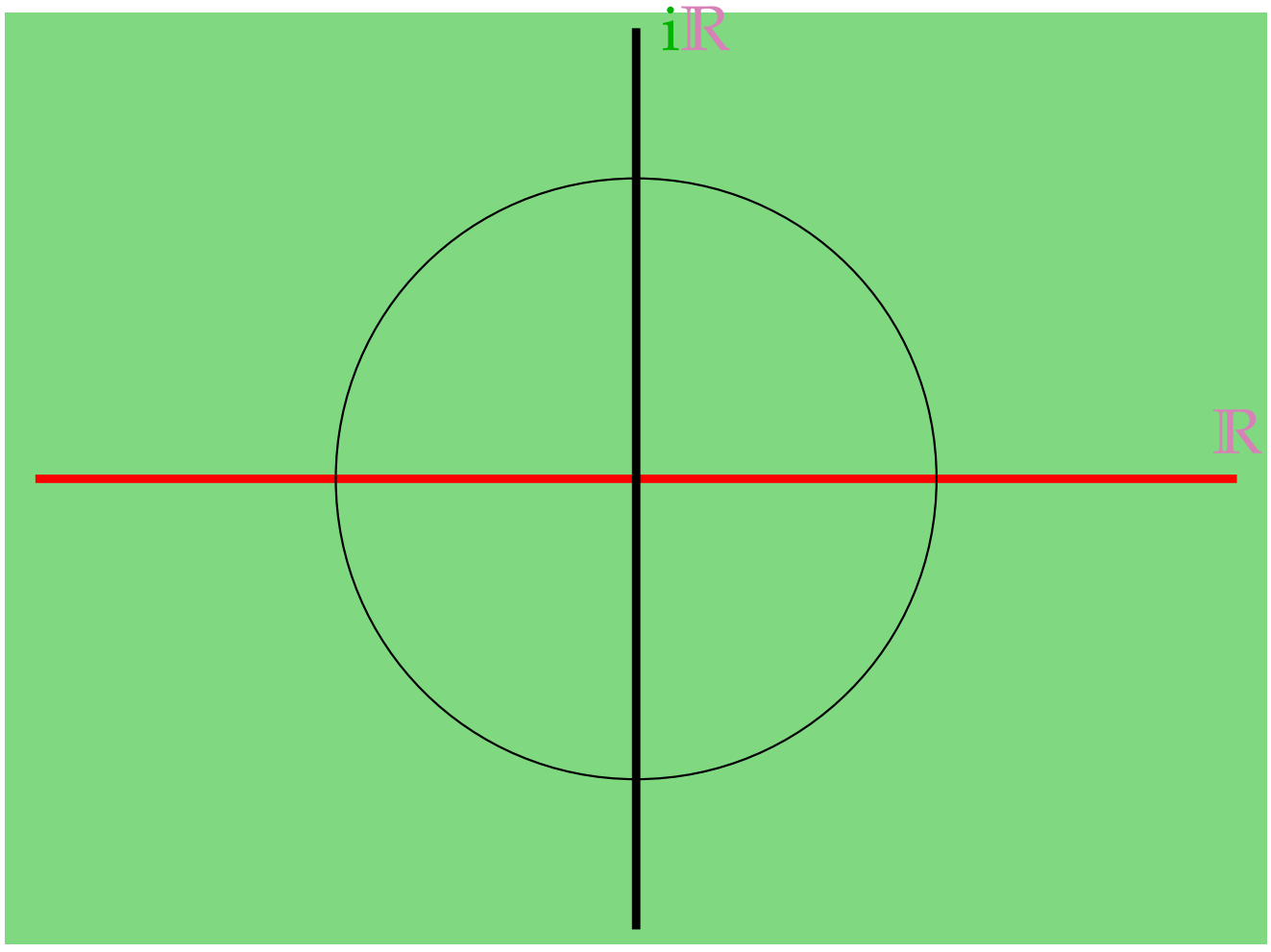

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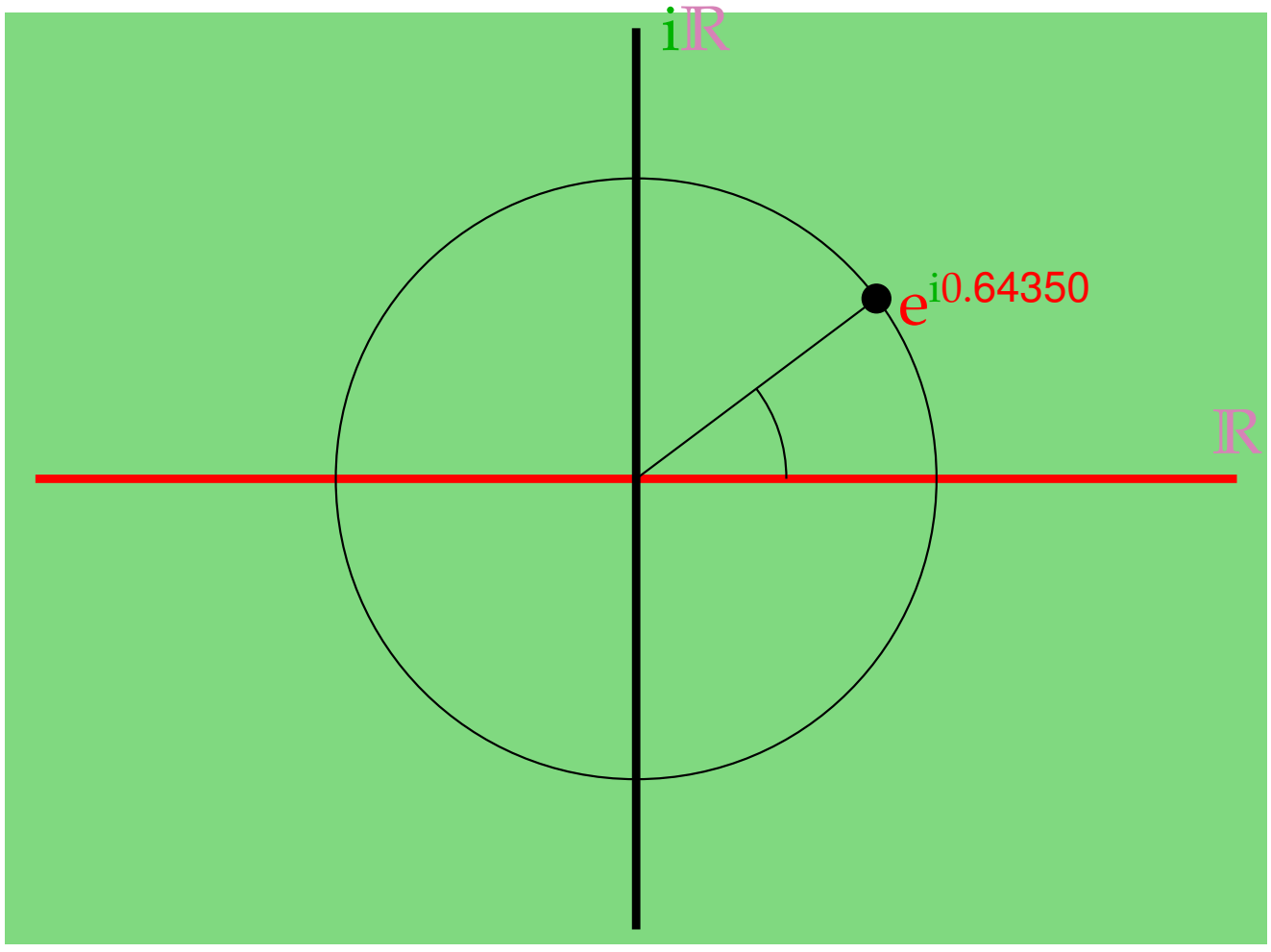

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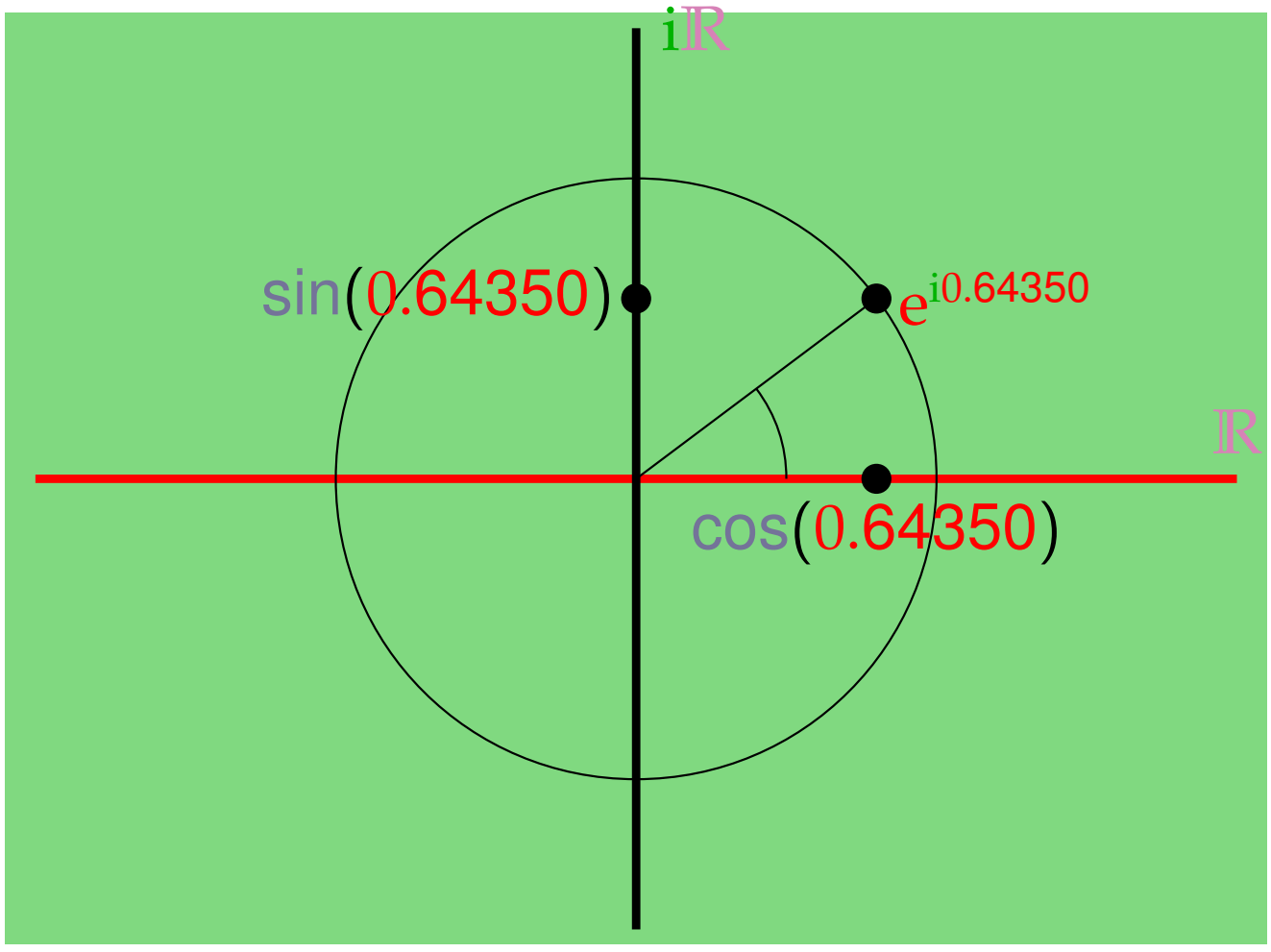

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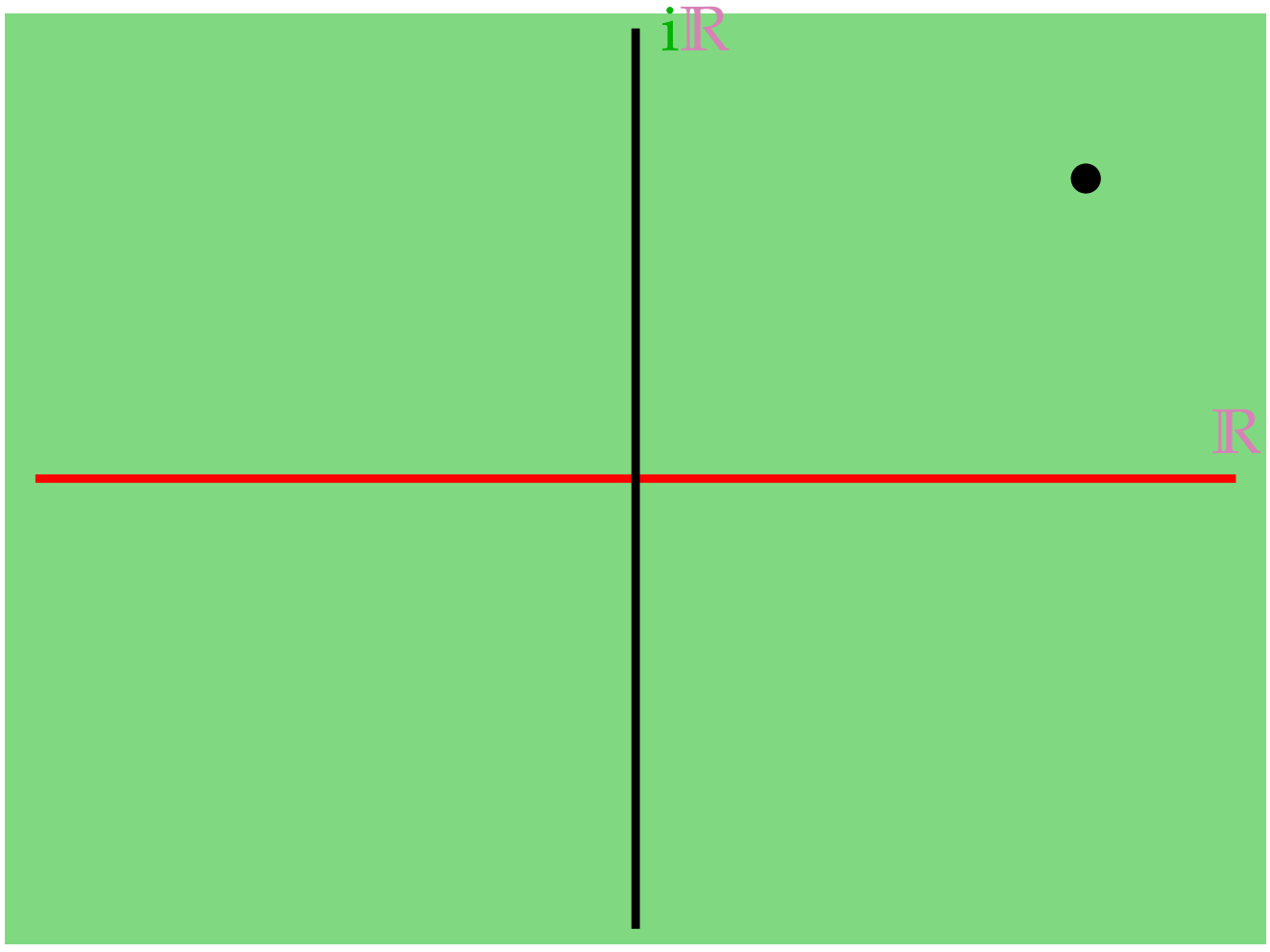

Absolute Value: $z = x + iy \mapsto |z| = \sqrt{x^2 + y^2}$

Argument: $z = x + iy \mapsto \arg(z) = \tan^{-1}(y/x)$

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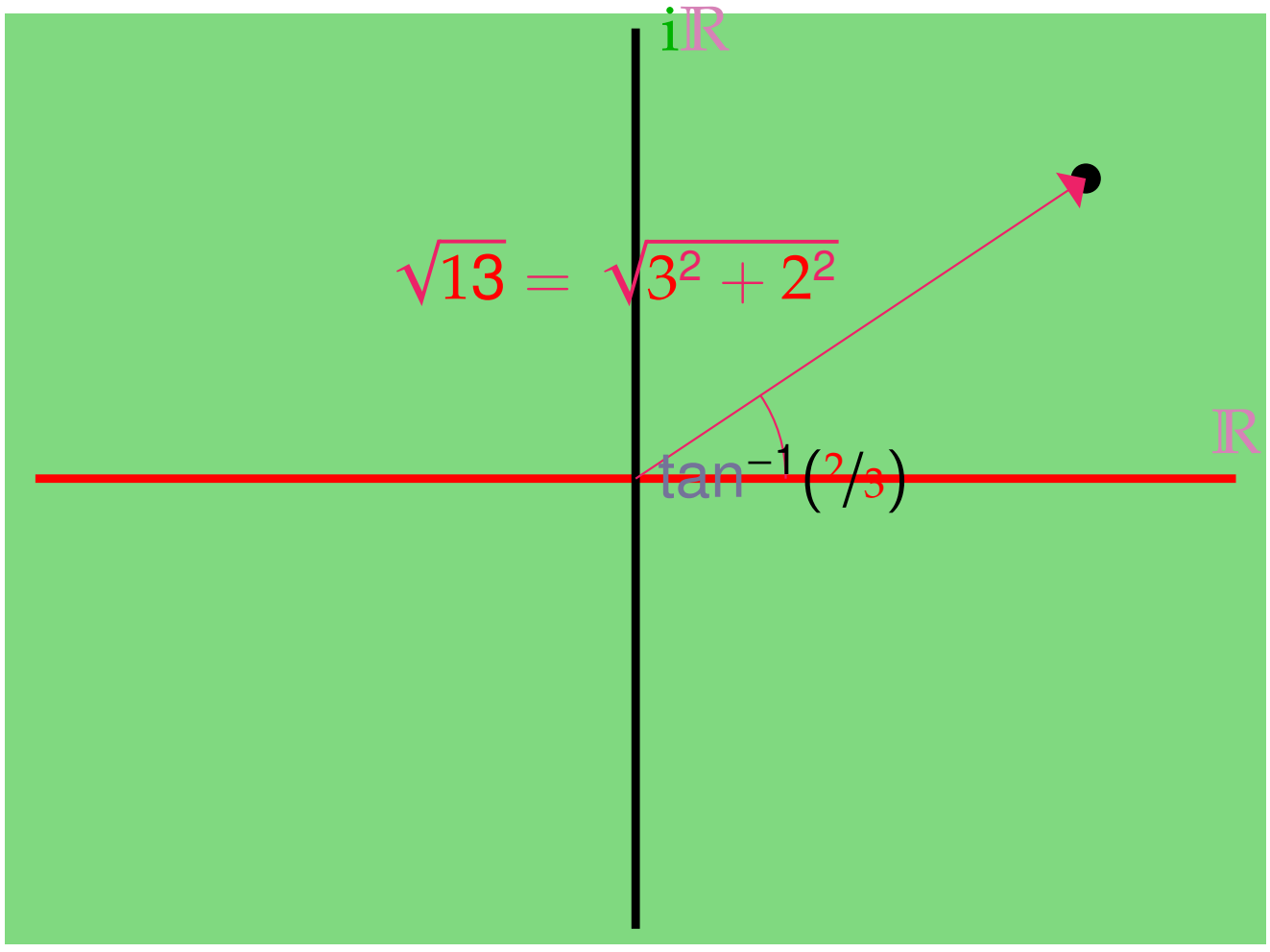

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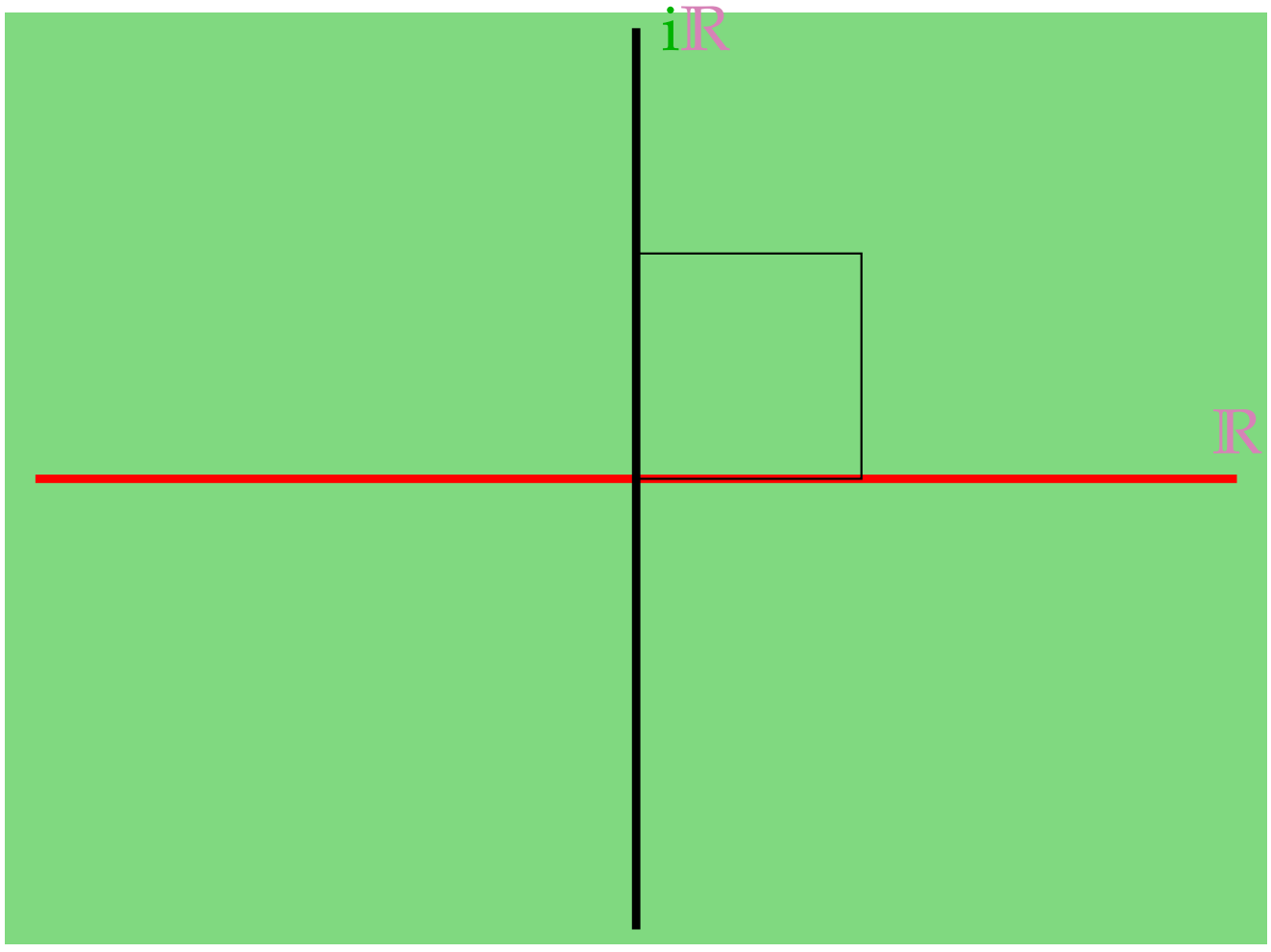
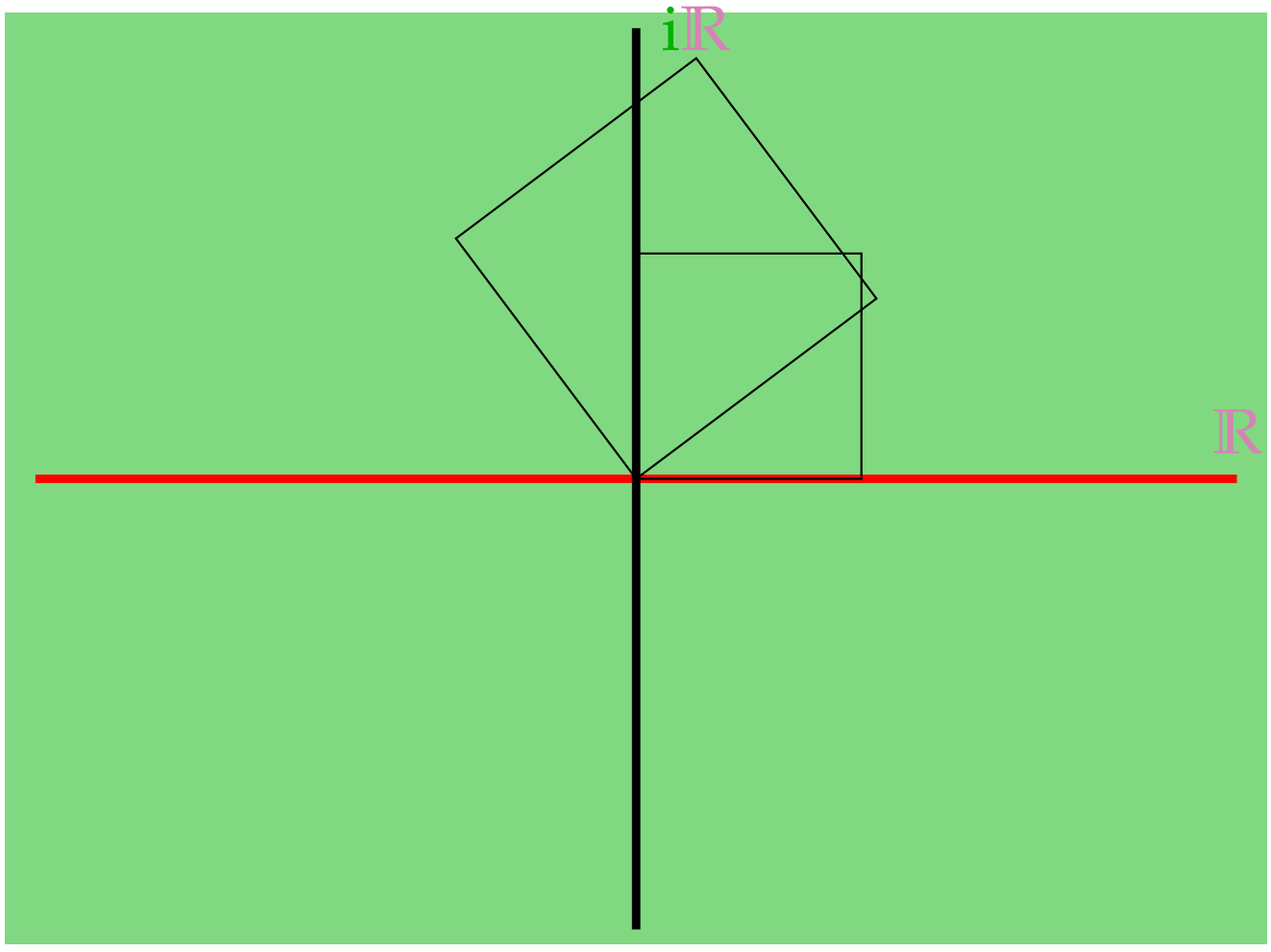


Multiplication = Rotation and Dilation

Pretty Pictures



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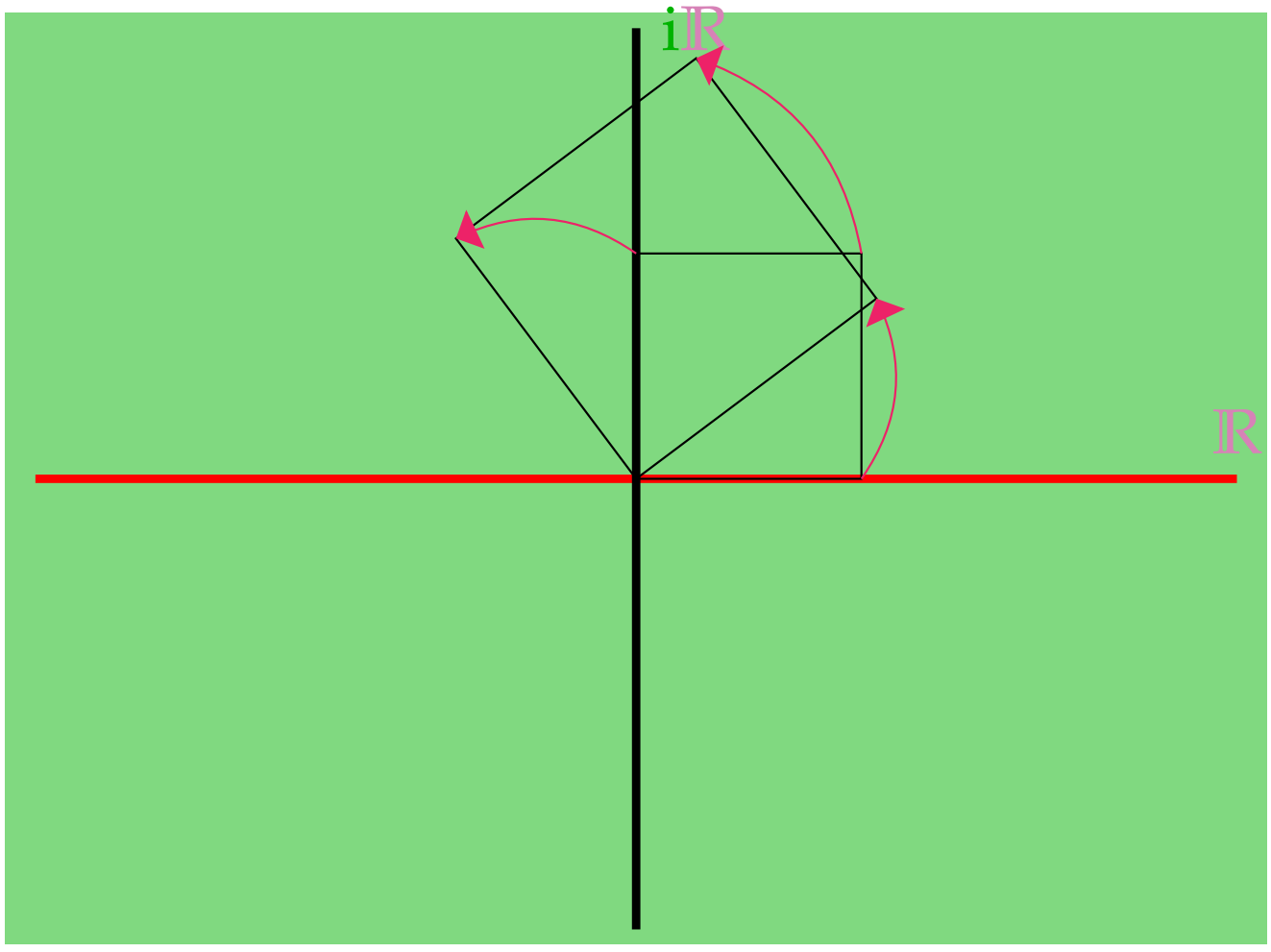

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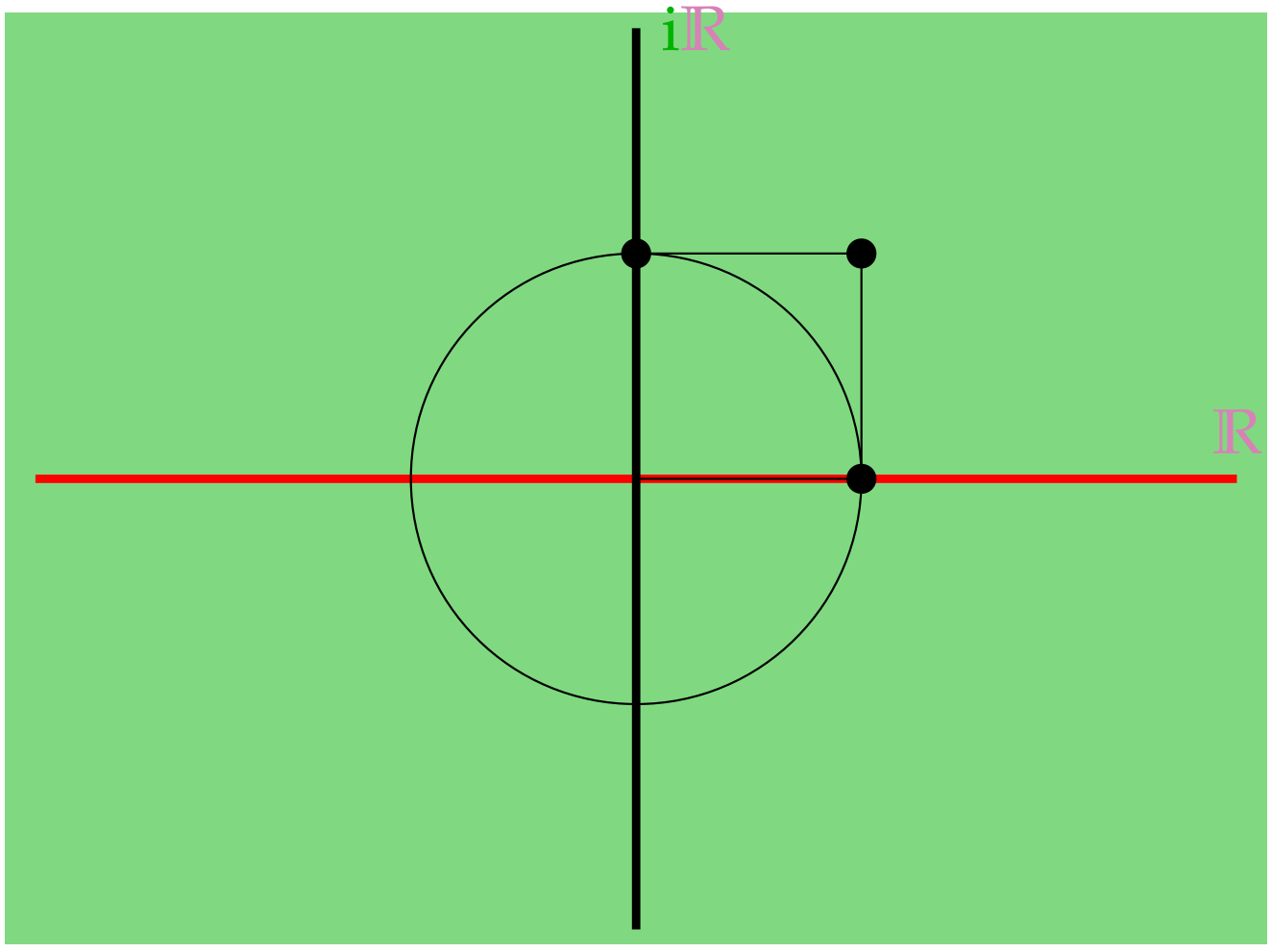

Reciprocal: $z \mapsto \frac{1}{z}$

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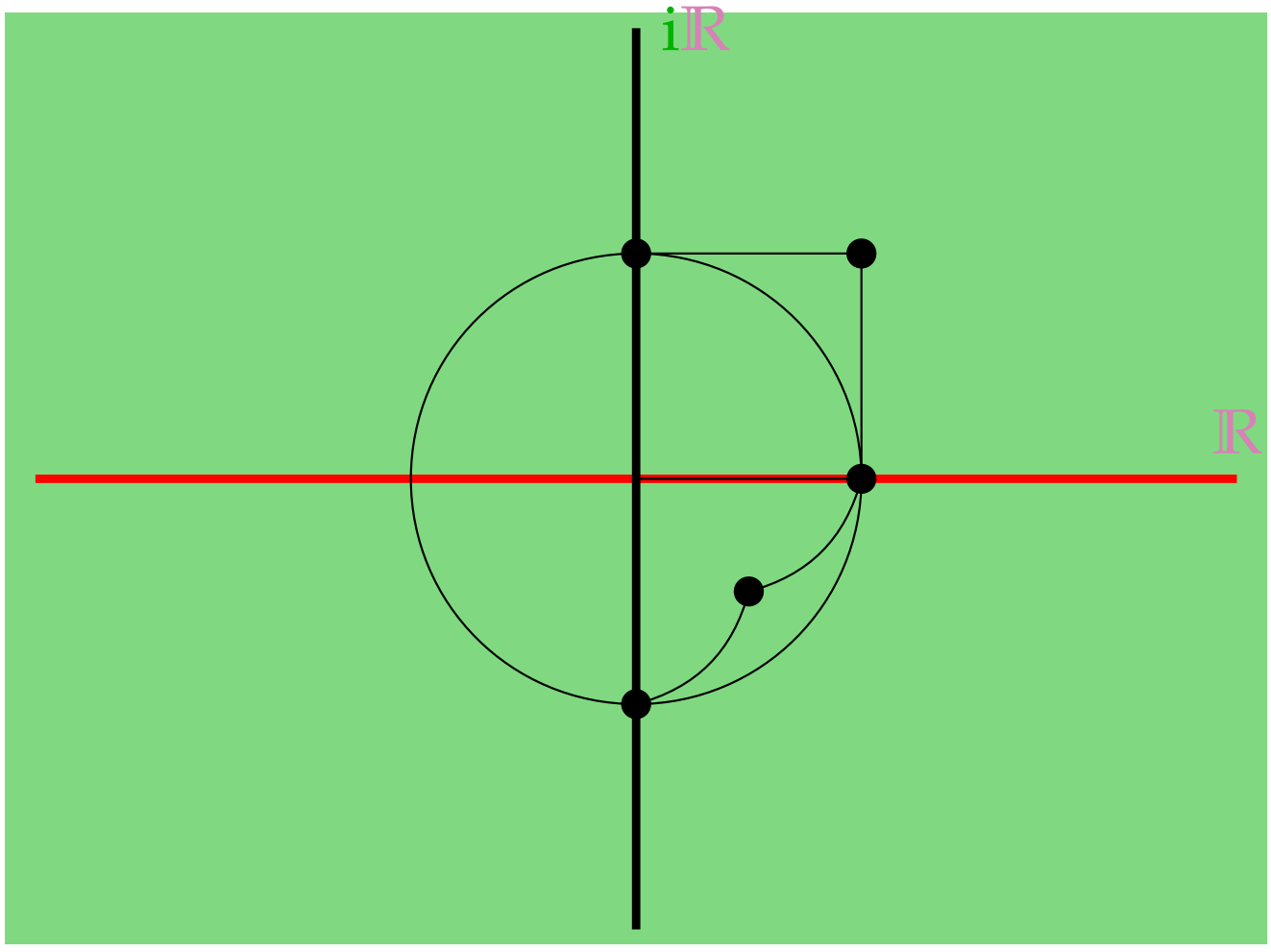

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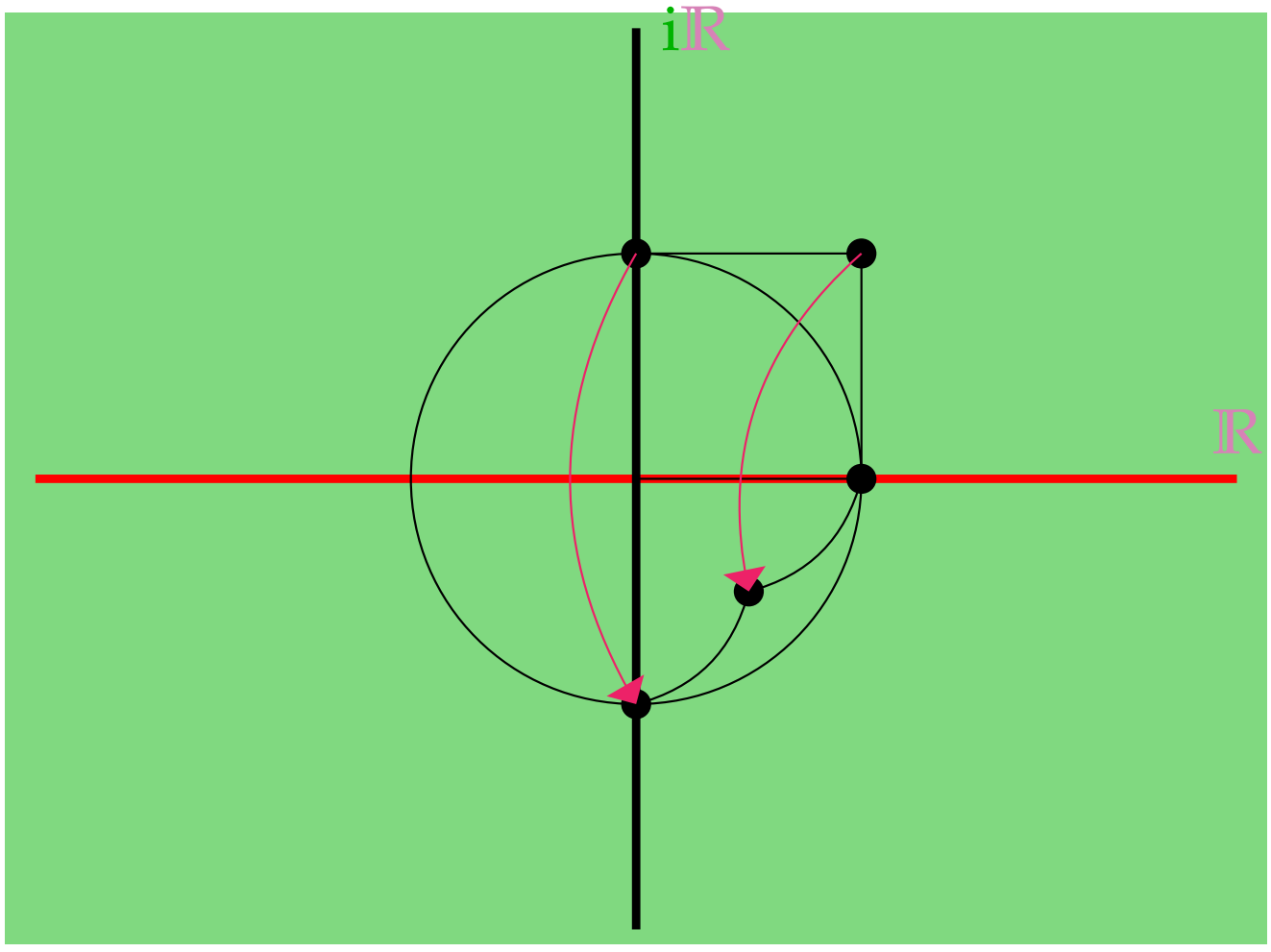

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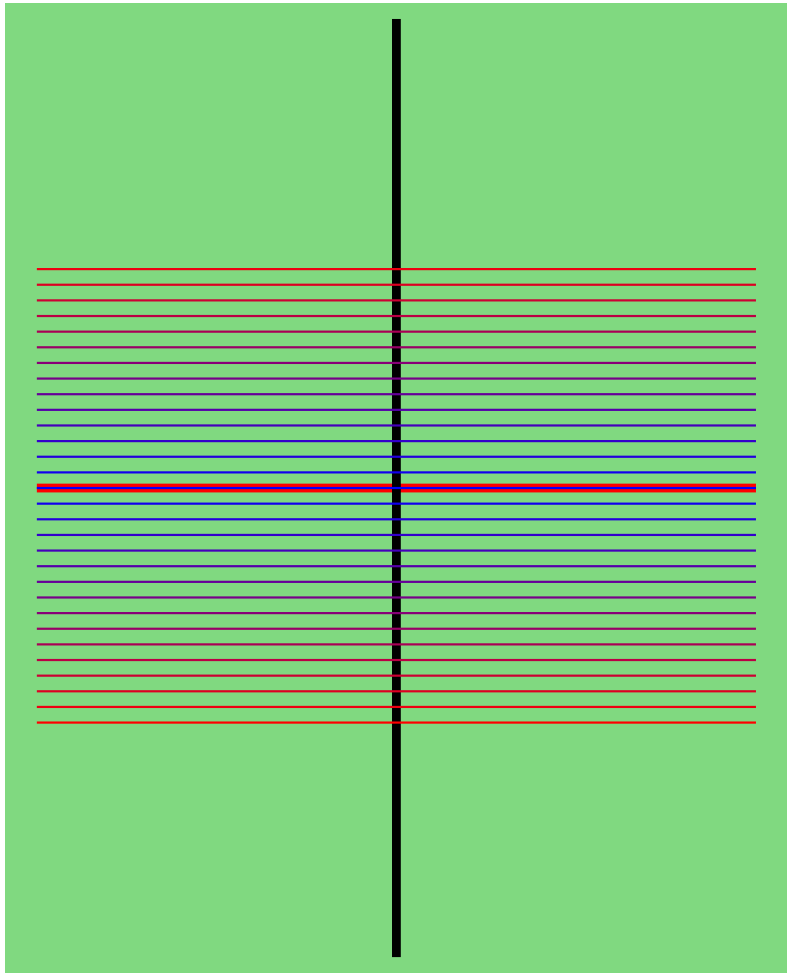

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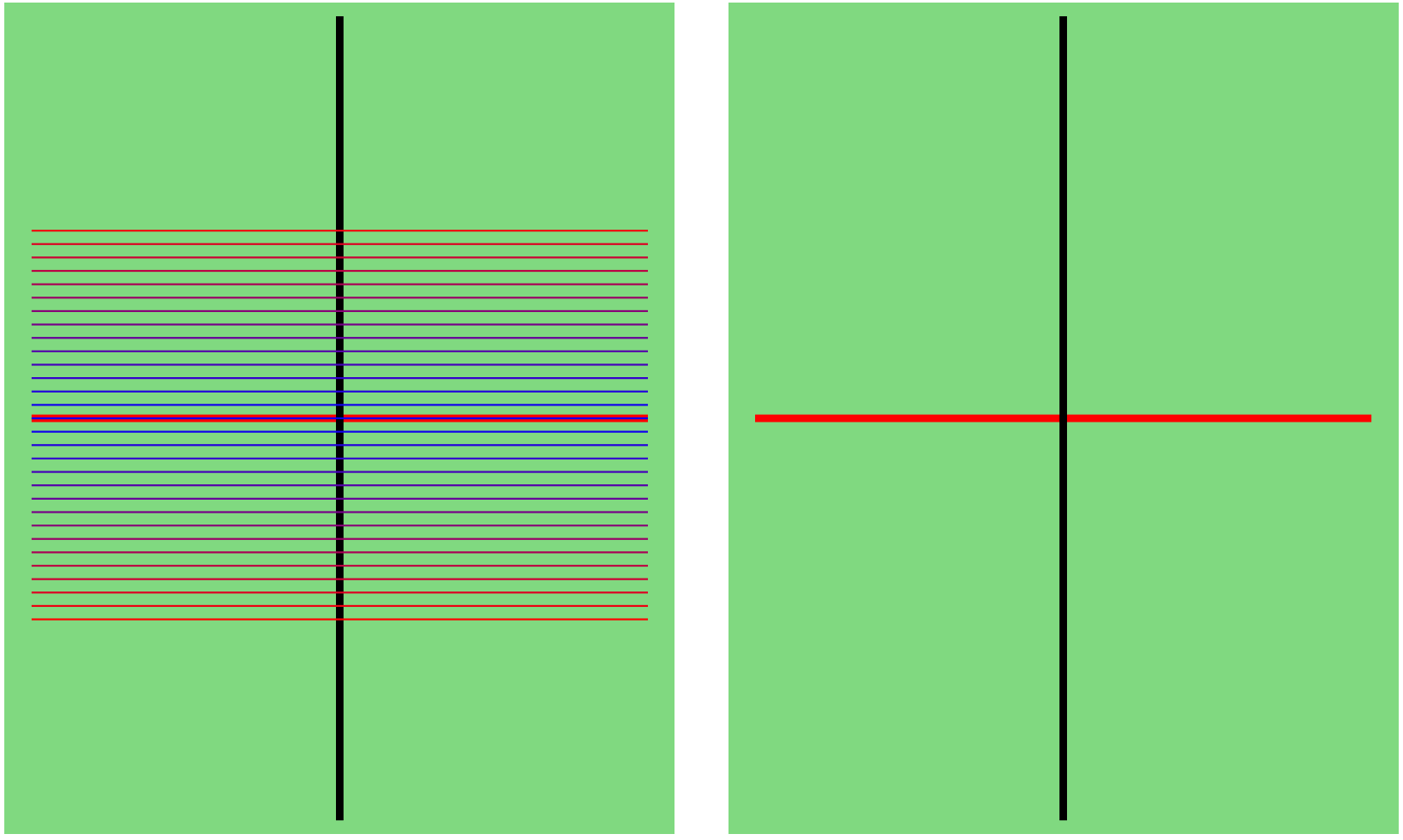
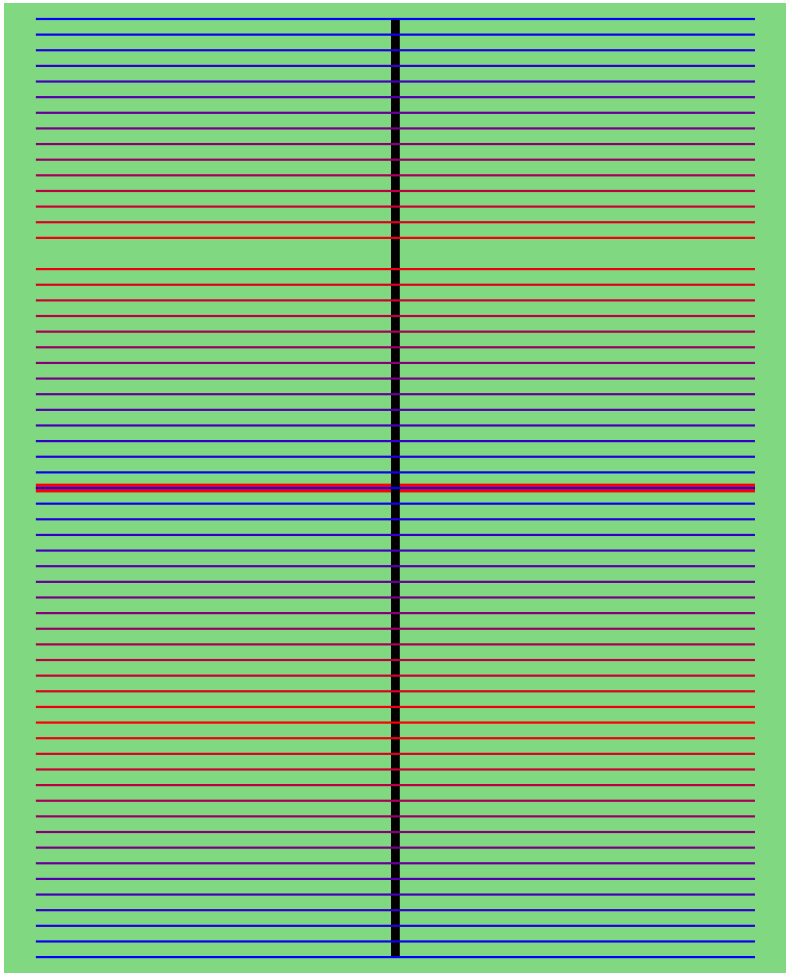
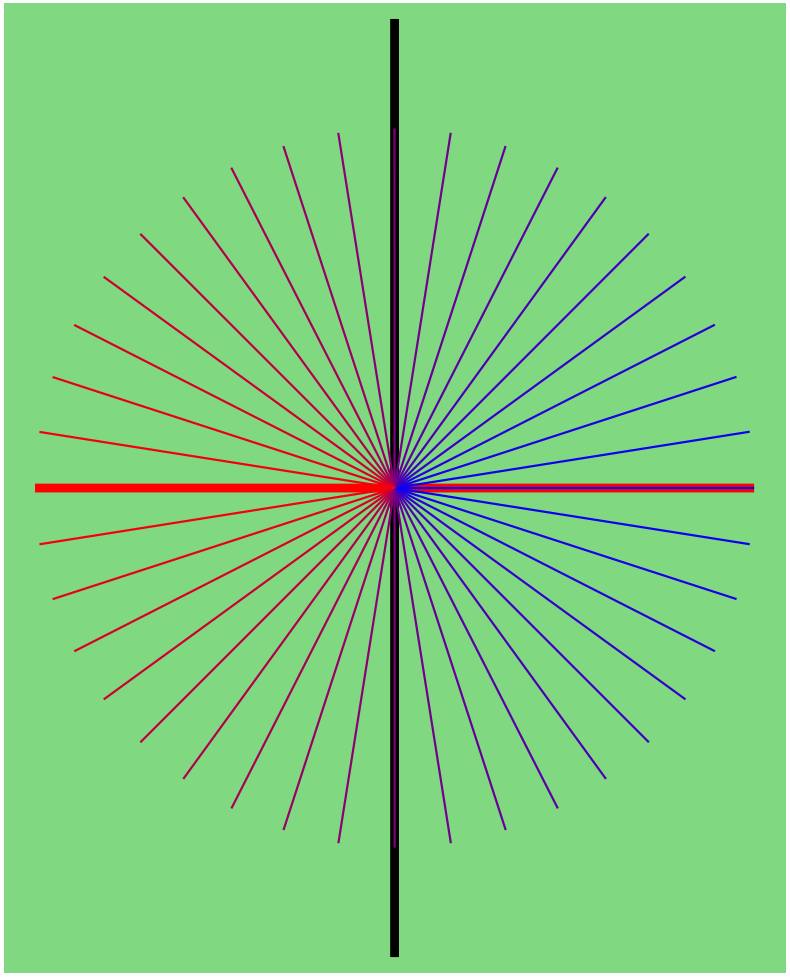


$z \mapsto e^z$

Not Just Pretty Pictures





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Back To Logarithms

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Powers

General Formula

$$z^w = e^{w \ln(z)}$$

Can be multi-valued!

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Special Case: Roots

$$\begin{aligned}
z^{1/m} &= e^{\ln(z)/m} \\
&= e^{(\ln(r)+i(\theta+2\pi n))/m} \\
&= e^{\ln(r)/m+i\theta/m+i2\pi n/m} \\
&= r^{1/m}e^{i\theta/m}e^{i2\pi n/m} \\
&= r^{1/m}\left(\cos(\theta/m + 2\pi n/m) + i\sin(\theta/m + 2\pi n/m)\right)
\end{aligned}$$

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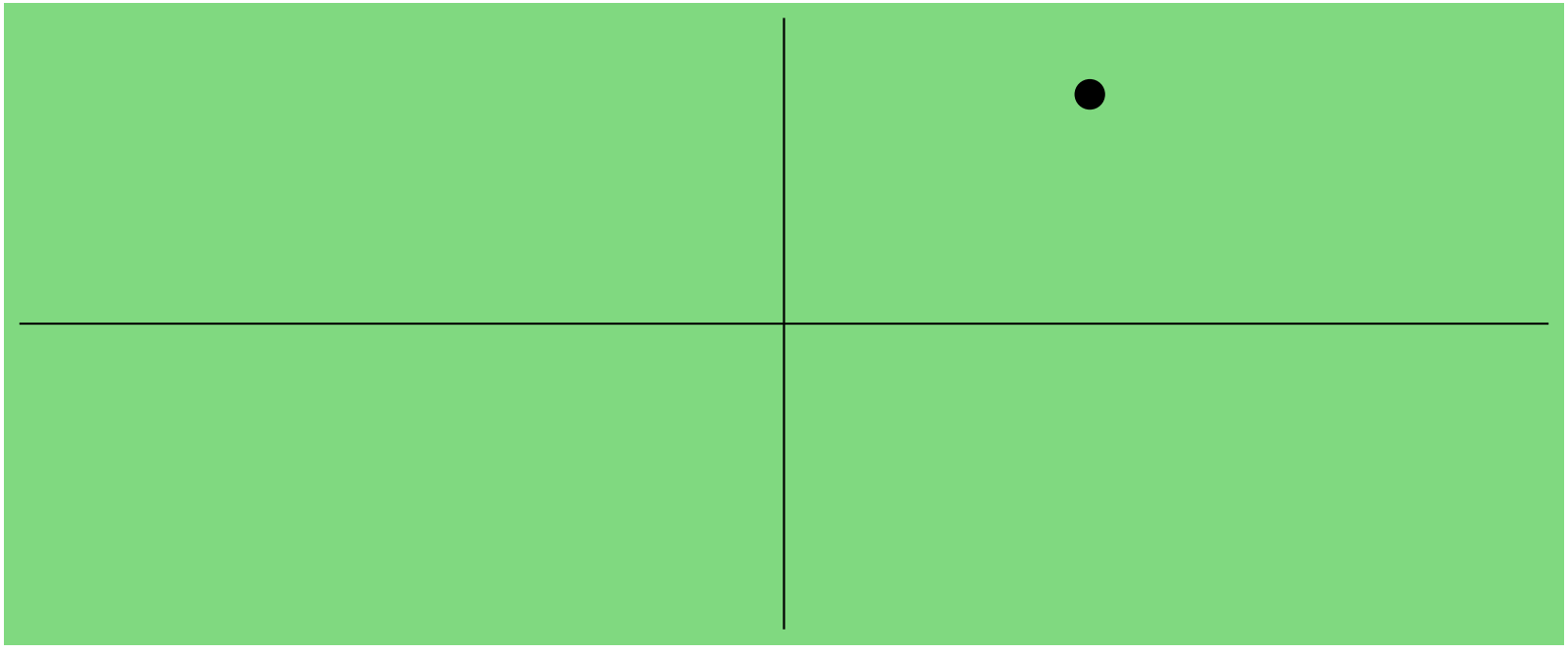
Note: Only m distinct values.

Example: 4th Roots of $4 + \mathrm{i}3$

$$r = \sqrt{4^2 + 3^2} = 5, \qquad \theta = \tan^{-1}(4/3) = 0.9273$$

$$\mapsto 5^{1/4} = 1.4953, \qquad \mapsto 0.23182$$



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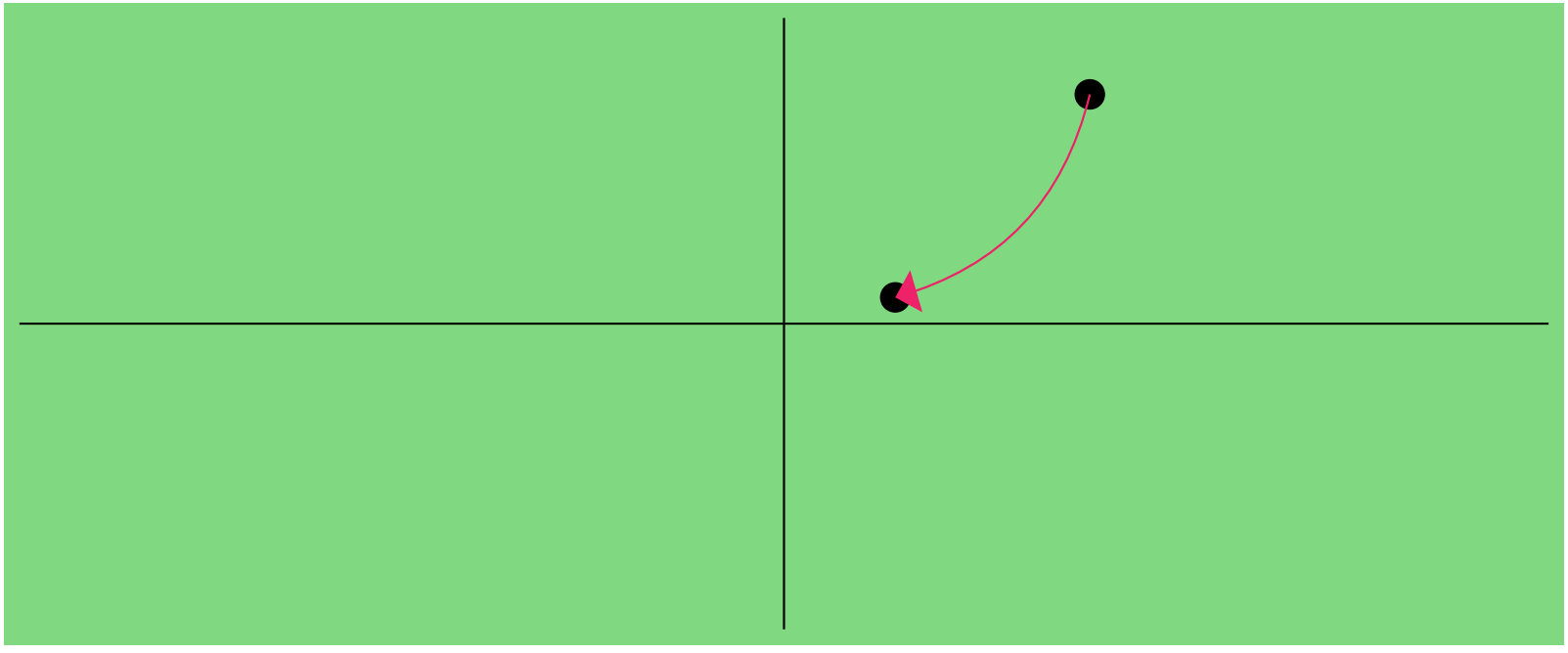

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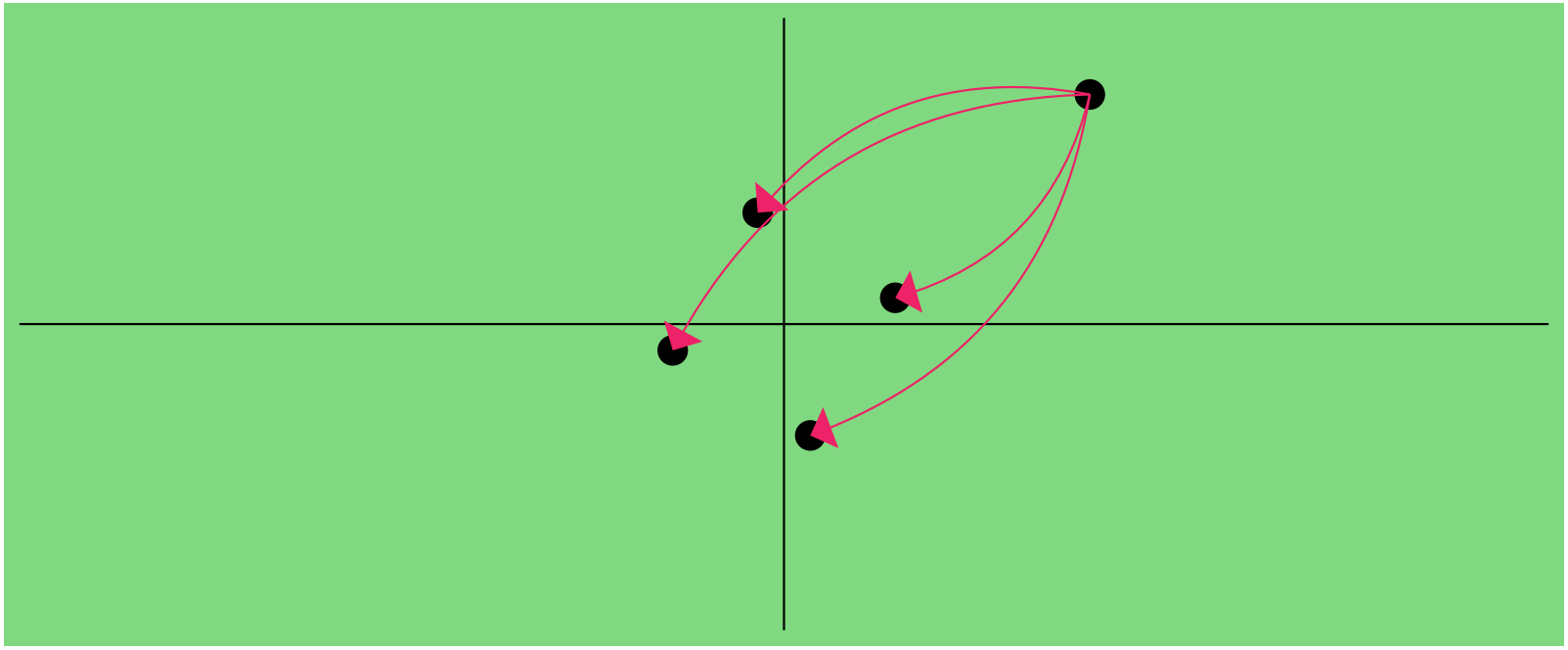
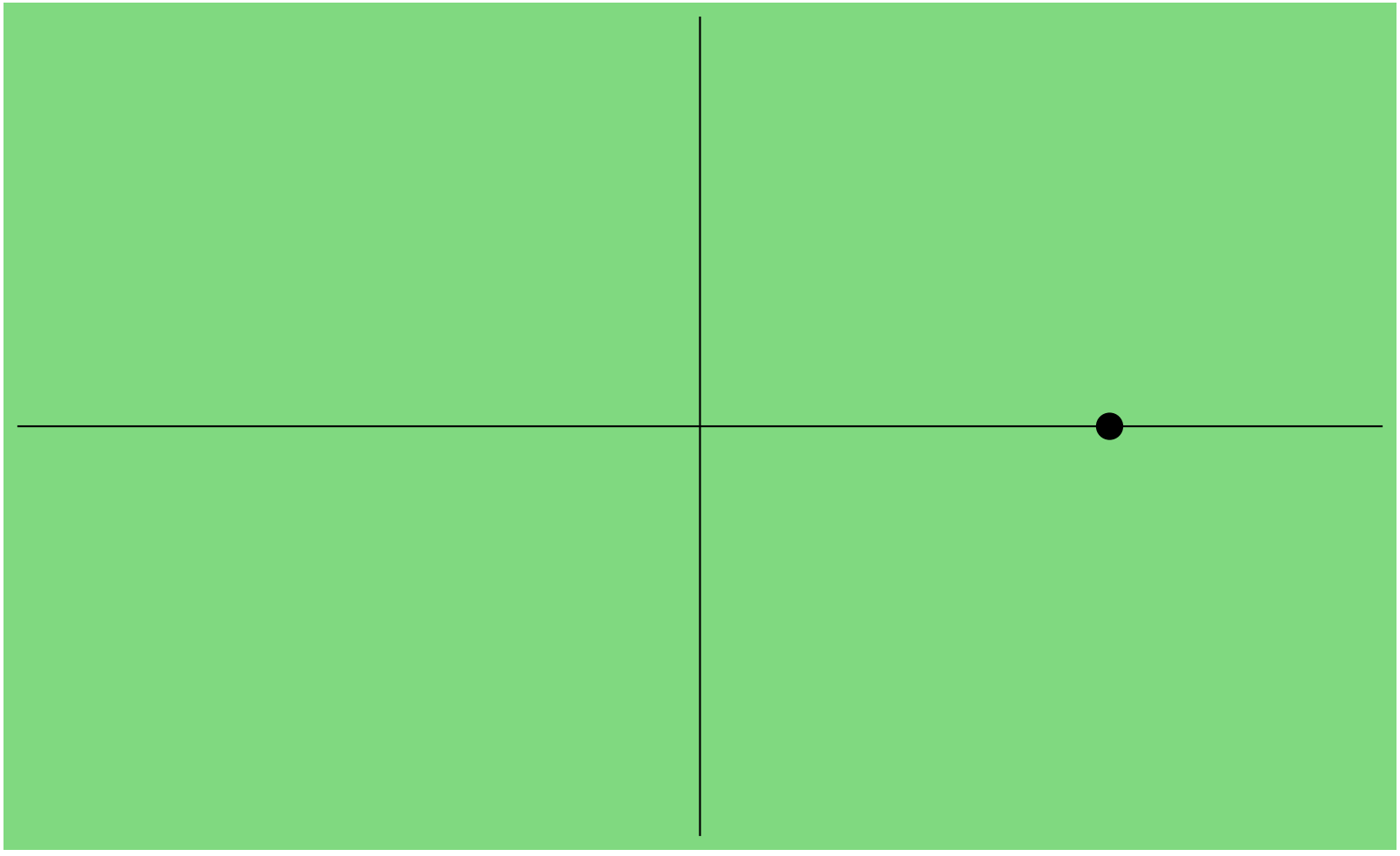


5th Roots of 1



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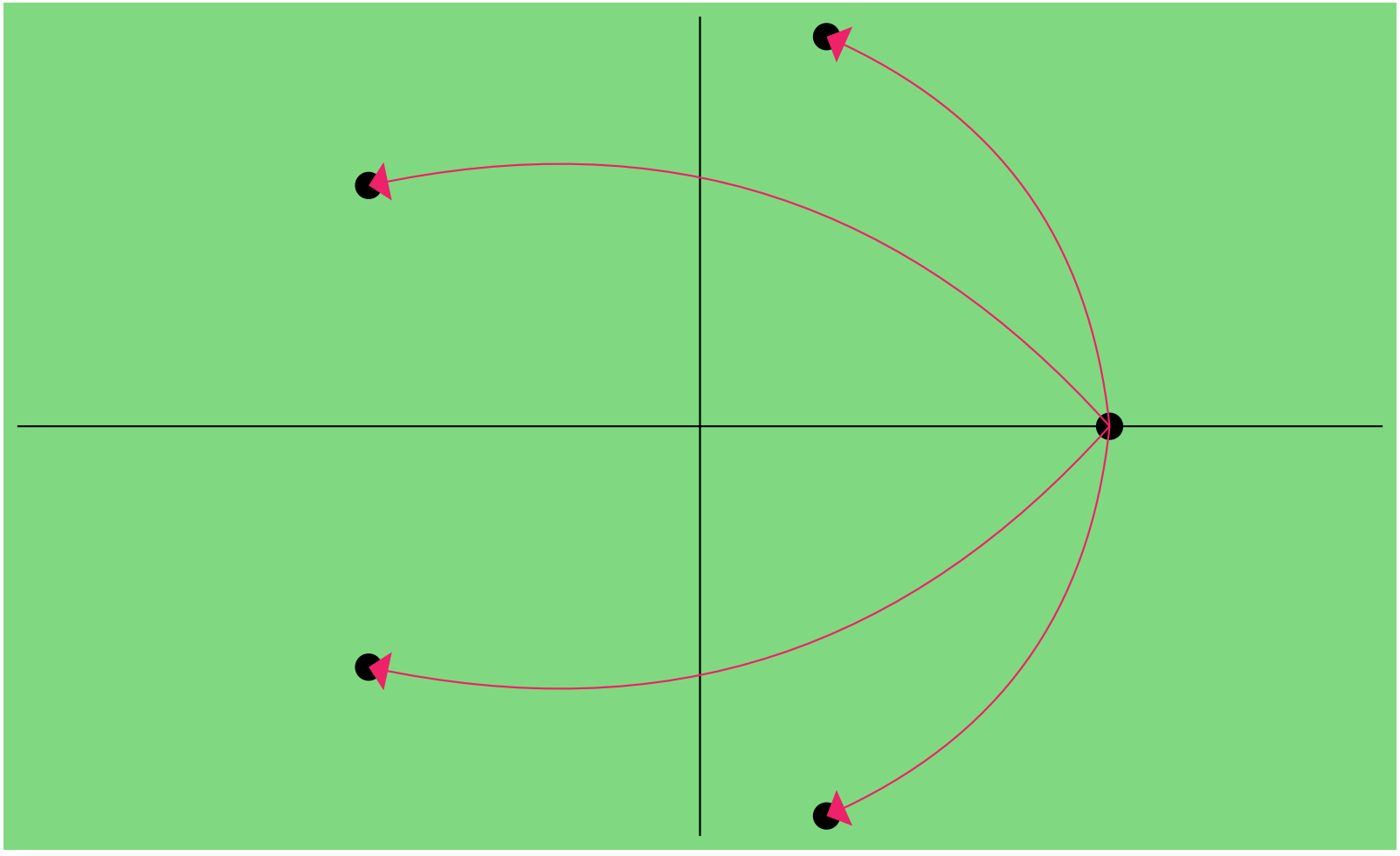

Summary

- The complex plane lets us see what's happening
- Can use knowledge of $\mathbb{R}^2$ to help understand $\mathbb{C}$
- e^z makes sense
- $\ln(z)$ makes sense — multi-valued!
- $z^{1/n}$ makes sense — multi-valued!
- These have nice geometric pictures.