TMA4115 Matematikk 3

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Lecture 1: y i?

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Introducing . . .

Introducing . . .

$$\sqrt{-1}$$

Usual Questions

Questions

Usual Questions

Questions

What?

Usual Questions

Questions

What?

Why?

Usual Questions

Questions

- What?
- Why?
- How?

Usual Questions

Questions

What? Square root of -1

Why?

How?

Usual Questions

Questions

What? Square root of -1

Why? Because it's useful

How?

Usual Questions

Questions

What?	Square root of -1
Why?	Because it's useful
How?	Ah, now that's an interesting question . . .

More detail on “why”

Mathematics is a language:

More detail on “why”

Mathematics is a language:

- Words $\longleftrightarrow$ Definitions

More detail on “why”

Mathematics is a language:

- Words $\longleftrightarrow$ Definitions
- Sentences $\longleftrightarrow$ Theorems

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(Roughly)

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Need words to express ideas

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Need words to express ideas
Words have no intrinsic reality;
their worth comes from their usefulness.

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- Does “dog” exist?

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- Does “dog” exist?
- Does “$\sqrt{-1}$” exist?

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(Roughly)

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- Does “dog” exist?
- Does “$\sqrt{-1}$” exist?

Wrong Question!

More detail on “why”

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(Roughly)

Need words to express ideas
Words have no intrinsic reality;
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- Is “dog” a useful concept?

- Is “$\sqrt{-1}$” a useful concept?

Right Question!

More detail on “why”

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- Words $\longleftrightarrow$ Definitions
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(Roughly)

Need words to express ideas
Words have no intrinsic reality;
their worth comes from their usefulness.

- Is “dog” a useful concept? Separates “dog food” from “cat food”
- Is “$\sqrt{-1}$” a useful concept?

Right Question!

Is $\sqrt{-1}$ Useful?

Real Question

What concepts does $\sqrt{-1}$ allow us to talk about?

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- Roots of any polynomial, e.g. $x^2 + 1$

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The shortest path between two truths in the real domain passes through the complex plane.

Hadamard

But How?

Complex Numbers

$=$

Real Numbers

$+$

$\sqrt{-1}$

But How?

Complex Numbers

$=$

Real Numbers

$+$

$\sqrt{-1}$

Is that it?

But How?

Complex Numbers

$=$

Real Numbers

$+$

$\sqrt{-1}$

Is that it?

Of course not!

Form Versus Function

Slogan

Mathematical objects are determined more by what they do than what they are.

Dynamic not static.

Form Versus Function

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What can we do with numbers?

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1. Add: $2 + 3 = 5$, $\pi + \mathrm{e} = 5.8599\ldots$

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1. Add: $2 + 3 = 5$, $\pi + e = 5.8599\ldots$
2. Subtract: $3 - 2 = 1$, $\pi - e = 0.42331\ldots$

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6. Subject to (lots of) rules.

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We'd like to do these with complex numbers.

Notational Pause

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Notation $\sqrt{-1}$ is unsafe!

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Set of Complex Numbers

Written as $\mathbb{C}$.

Generating Complex Numbers

- Start: $\mathbb{R} \subseteq \mathbb{C}$ and $\mathrm{i} \in \mathbb{C}$.

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Story So Far

$$x, y \in \mathbb{R} \Longrightarrow x + \mathrm{i}y \in \mathbb{C}$$

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Story So Far

$$x, y \in \mathbb{R} \Longrightarrow x + \mathrm{i}y \in \mathbb{C}$$

Examples

1. $2 + \mathrm{i}3$
2. $1 + \mathrm{i}(-1)$
3. $\pi + \mathrm{i}e$

What Else?

Question

What else must we have?
(So that $+$, $-$, $\times$, $\div$ defined)

What Else?

Question

What else must we have?
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Answer

Nothing!

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Answer

Nothing!
Because of the rules.

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Question

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Answer

Nothing!
Because of the rules.

Rules? What rules?

Same as $\mathbb{R}$.

Addition

$$
\begin{aligned}
(\mathrm{e} + \mathrm{i}3) + (\pi + \mathrm{i}2) &= (\mathrm{e} + \mathrm{i}3) + (\mathrm{i}2 + \pi) \\
&= \mathrm{e} + (\mathrm{i}3 + (\mathrm{i}2 + \pi)) \\
&= \mathrm{e} + ((\mathrm{i}3 + \mathrm{i}2) + \pi) \\
&= \mathrm{e} + (\pi + \mathrm{i}(3 + 2)) \\
&= (\mathrm{e} + \pi) + \mathrm{i}5
\end{aligned}
$$

Addition

$$\begin{aligned}(\mathrm{e}+\mathrm{i}3)+(\pi+\mathrm{i}2) &= (\mathrm{e}+\mathrm{i}3)+(\mathrm{i}2+\pi)\\ &= \mathrm{e}+(\mathrm{i}3+(\mathrm{i}2+\pi))\\ &= \mathrm{e}+((\mathrm{i}3+\mathrm{i}2)+\pi)\\ &= \mathrm{e}+(\pi+\mathrm{i}(3+2))\\ &= (\mathrm{e}+\pi)+\mathrm{i}5\end{aligned}$$

Of the form $x + \mathrm{i}y$ with $x, y \in \mathbb{R}$.

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Of the form $x+\mathrm{i}y$ with $x, y \in \mathbb{R}$.

General Rule

$$(x+\mathrm{i}y)+(u+\mathrm{i}v) = (x+u)+\mathrm{i}(y+v)$$

Subtraction

$$\begin{aligned}(\mathrm{e}+\mathrm{i}3)-(\pi+\mathrm{i}2) &= (\mathrm{e}+\mathrm{i}3)+((-\pi)+\mathrm{i}(-2)) \\ &= (\mathrm{e}-\pi)+\mathrm{i}(3-2) \\ &= (\mathrm{e}-\pi)+\mathrm{i}1 \\ &= (\mathrm{e}-\pi)+\mathrm{i}\end{aligned}$$

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Of the form $x + \mathrm{i}y$ with $x, y \in \mathbb{R}$.

General Rule

$$(x+\mathrm{i}y)-(u+\mathrm{i}v) = (x-u)+\mathrm{i}(y-v)$$

Multiplication

$$
\begin{aligned}
(\mathrm{e}+\mathrm{i}3)(\pi+\mathrm{i}2) &= \mathrm{e}\pi+\mathrm{e}\mathrm{i}2+\mathrm{i}3\pi+\mathrm{i}3\mathrm{i}2 \\
&= \mathrm{e}\pi+\mathrm{i}\mathrm{e}2+\mathrm{i}3\pi+\mathrm{i}^2 6 \\
&= \mathrm{e}\pi+\mathrm{i}(\mathrm{e}2+3\pi)+\mathrm{i}^2 6 \\
&= \mathrm{e}\pi+\mathrm{i}(\mathrm{e}2+3\pi)+-1\times 6 \\
&= (\mathrm{e}\pi-6)+\mathrm{i}(\mathrm{e}2+3\pi)
\end{aligned}
$$

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&= (\mathrm{e}\pi-6)+\mathrm{i}(\mathrm{e}2+3\pi)
\end{aligned}
$$

General Rule

$$(x+\mathrm{i}y)(u+\mathrm{i}v) = (xu-yv)+\mathrm{i}(xv+yu)$$

Division

Question

What is division?

Division

Question

What is division?

Answer

$$\frac{1}{x} \times x = 1$$

Division

Question

What is division?

Answer

$$\frac{1}{x} \times x = 1$$

Question

What is 1?

Division

Question

What is division?

Answer

$$\frac{1}{x} \times x = 1$$

Question

What is 1?

Answer

$$1 \times x = x$$

Division

- Finding 1: solve

$$(x + \mathrm{i}y)(u + \mathrm{i}v) = (u + \mathrm{i}v)$$

Division

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- Specific instance: solve

$$(x + \mathrm{i}y)(2 + \mathrm{i}3) = 2 + \mathrm{i}3$$

Division

- Finding 1: solve

$$(x + \mathrm{i}y)(u + \mathrm{i}v) = (u + \mathrm{i}v)$$

- Specific instance: solve

$$(x + \mathrm{i}y)(2 + \mathrm{i}3) = 2 + \mathrm{i}3$$

- Expand out multiplication:

$$2x - 3y + \mathrm{i}(2y + 3x) = 2 + \mathrm{i}3$$

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- Key Fact: $u + \mathrm{i}v = s + \mathrm{i}t \iff u = s$ and $v = t$

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- Key Fact: $u + \mathrm{i}v = s + \mathrm{i}t \iff u = s$ and $v = t$
- Solve: $2x - 3y = 2$ and $2y + 3x = 3$

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- Key Fact: $u + \mathrm{i}v = s + \mathrm{i}t \iff u = s$ and $v = t$
- Solve: $2x - 3y = 2$ and $2y + 3x = 3$
- Solution: $x = 1$, $y = 0$.

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- Key Fact: $u + \mathrm{i}v = s + \mathrm{i}t \iff u = s$ and $v = t$
- Solve: $2x - 3y = 2$ and $2y + 3x = 3$
- Solution: $x = 1$, $y = 0$.
- Conclusion: $1 = 1 + \mathrm{i}0$ (unsurprisingly)

Division

- Finding $1/(x + \mathrm{i}y)$: solve

$$(x + \mathrm{i}y)(u + \mathrm{i}v) = 1 + \mathrm{i}0$$

Division

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- Specific instance: solve

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- Solve: $2u - 3v = 1$ and $2v + 3u = 0$
- Substitute: $v = -3u/2$

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- Solve: $2u - 3v = 1$ and $2v + 3u = 0$
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- Get: $u(2 + 9/2) = 1$

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- Finding $1/(x + \mathrm{i}y)$: solve

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- Solve: $2u - 3v = 1$ and $2v + 3u = 0$
- Substitute: $v = -3u/2$
- Get: $u(2 + 9/2) = 1$
- Solution: $u = 2/13$, $v = -3/13$.

Division

General Formula

$x + \mathrm{i}y \neq 0$ then

$$\frac{1}{x + \mathrm{i}y} = \frac{x}{x^2 + y^2} - \mathrm{i}\frac{y}{x^2 + y^2}$$

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General Formula

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Looks a bit scary . . . split it up:

Division

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$x + \mathrm{i}y \neq 0$ then

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Looks a bit scary . . . split it up:

$$\frac{1}{x + \mathrm{i}y} = \frac{1}{x^2 + y^2}(x - \mathrm{i}y)$$

Story So Far

Complex Numbers

- Notation $\mathbb{C}$

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- Same rules as in $\mathbb{R}$ with . . .

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- $\mathbb{R} \subseteq \mathbb{C}$ as $x \mapsto x + \mathrm{i}0$

Story So Far

Complex Numbers

- Notation $\mathbb{C}$
- $z \in \mathbb{C}$ represented as $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$
- Add, Subtract, Multiply, Divide
- Same rules as in $\mathbb{R}$ with . . .
- $\mathrm{i}^2 = -1$
- $\mathbb{R} \subseteq \mathbb{C}$ as $x \mapsto x + \mathrm{i}0$
- Conventions: leave out 1 or 0 if unambiguous:

$$2 + \mathrm{i}0 \to 2, \qquad 2 + \mathrm{i}1 \to 2 + \mathrm{i}$$

Complex Conjugation

Division formula:

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Summary

- Complex numbers are like real numbers with $\mathbf{i}$
- Add, Subtract, Multiply, Divide just as in $\mathbb{R}$
- New thing: complex conjugation

Things To Ponder

- Can we compare complex numbers? $z \leq w$?
- Gaussian integers: all complex numbers of form $n + \mathbf{i}m$. How does this compare to $\mathbb{Z}$?