# Testing Linear Independence

## Tests

Solutions $y_1$ and $y_2$

1. Is $y_1 = 0$?
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
3. Is $y_1'(x_0) \neq 0$ and $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1$?

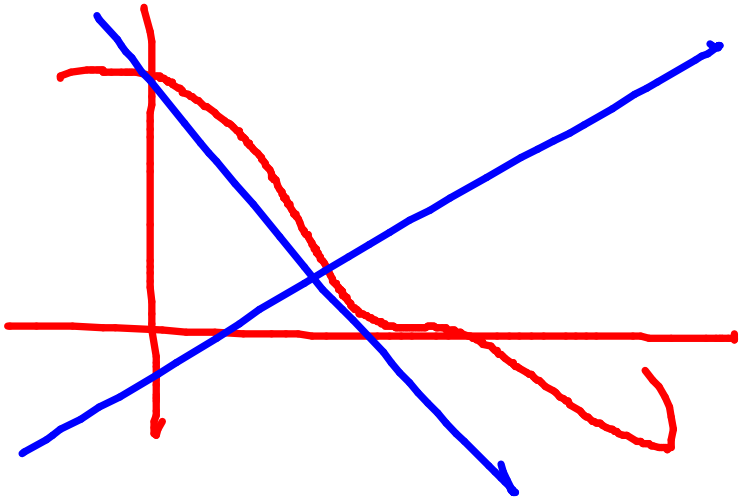
$ay_1 + by_2 = 0$

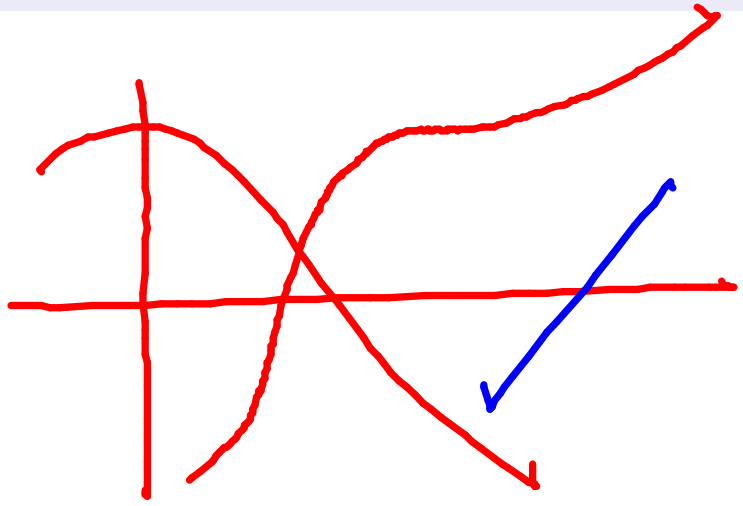
$\Rightarrow a = b = 0$

$y \neq 0$ $\quad y_2 = \frac{a}{b} y_1$

Yes to any $\Longrightarrow$ dependent

No to all $\Longrightarrow$ independent





# Examples

$ay_1 + by_2 = 0$?

$$y'' - 2y' + y = 0$$

Solutions: $y_2(x) = e^x$, $y_1(x) = xe^x$
Question: Are these linearly independent?

1. Is $y_1 = 0$?
   No. $2e^2 \neq 0$.
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
   E.g. $x_0 = 1$: "Yes ($1e^1 \neq 0$) and No ($e^x \neq xe^x$)".

# Examples

$$y'' - 3x^{-1}y' + 3x^{-2}y = 0$$

Solutions: $y_1(x) = x$, $y_2(x) = x^3$
Question: Are these linearly independent?

1. Is $y_1 = 0$?
   No. $1 \neq 0$.
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
   E.g. $x_0 = 1$: "Yes ($1 \neq 0$) and No ($8 \neq 2$)".
3. Is $y_1'(x_0) \neq 0$ and $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1$?
   E.g. ($x_0 = 1$): "Yes ($1 \neq 0$) and No ($x^3 \neq 3x$)".

# Examples

$$y'' - 3x^{-1}y' + 3x^{-2}y = 0$$

Solutions: $y_1(x) = x$, $y_2(x) = x^3$
Question: Are these linearly independent?

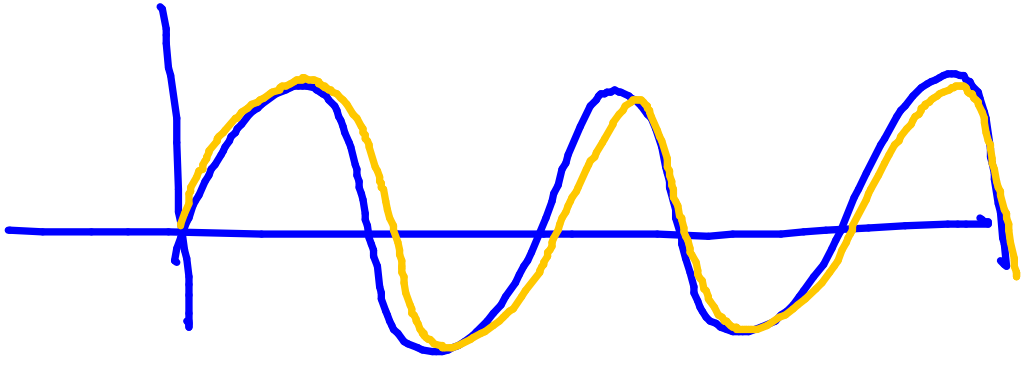
1. Is $y_1 = 0$?
   No. $1 \neq 0$.
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
   E.g. $x_0 = 1$: "Yes ($1 \neq 0$) and No ($8 \neq 2$)".
3. Is $y_1'(x_0) \neq 0$ and $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1$?
   E.g. ($x_0 = 1$): "Yes ($1 \neq 0$) and No ($3 \neq 6$)".

# Examples

$$y'' + 4y = 0$$

Solutions: $y_1(x) = \sin(2x)$, $y_2(x) = \sin(x)\cos(x)$
Question: Are these linearly independent?

1. Is $y_1 = 0$?
   No. $1 = \sin(2\pi/4) \neq 0$.
2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?
   E.g. $x_0 = \pi/4$: "Yes ($1 \neq 0$) and Yes ($\sin(x)\cos(x) = 1/2 \sin(2x)$)".



# Testing Linear Independence

1. Is $y_1 = 0$?

2. Is $y_1(x_0) \neq 0$ and $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1$?

   $y_3 := y_2 - \frac{y_2(x_0)}{y_1(x_0)} y_1$ solution with $y_3(x_0) = 0$

   $y_3 = 0 \iff y_3'(x_0) = 0$

   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y_2'(x_0) = \frac{y_2(x_0)}{y_1(x_0)} y_1'(x_0)$

   $y_2 = \frac{y_2(x_0)}{y_1(x_0)} y_1 \iff y_2'(x_0) y_1(x_0) - y_2(x_0) y_1'(x_0) = 0$

3. Is $y_1'(x_0) \neq 0$ and $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1$?

   $y_3 := y_2 - \frac{y_2'(x_0)}{y_1'(x_0)} y_1$ solution with $y_3'(x_0) = 0$

   $y_3 = 0 \iff y_3(x_0) = 0$

   $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1 \iff y_2(x_0) = \frac{y_2'(x_0)}{y_1'(x_0)} y_1(x_0)$

   $y_2 = \frac{y_2'(x_0)}{y_1'(x_0)} y_1 \iff y_2(x_0) y_1'(x_0) - y_2'(x_0) y_1(x_0) = 0$

# Testing Linear Independence

## Claim

$y_1$ and $y_2$ not linearly independent

$$\Longleftrightarrow$$

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) = 0$$

for some $x_0$

How can it be zero?

1. $y_1(x_0) = 0$ and $y_1'(x_0) = 0 \implies$ test 1 holds
2. $y_1(x_0) \neq 0 \implies$ test 2 holds
3. $y_1'(x_0) \neq 0 \implies$ test 3 holds

$ay_1 + by_2 = 0 \implies a = b = 0$?

# Wronskian and Linear Independence

$$y'' + p(x)y' + q(x)y = 0$$

$y_1$ and $y_2$ solutions of ODE.

$y_1$, $y_2$ linearly independent

$$\Longleftrightarrow$$

$W(y_1, y_2)(x) \neq 0$ for all $x \in I$

$$\Longleftrightarrow$$

$W(y_1, y_2)(x_0) \neq 0$ for some $x_0 \in I$

# The Norwegian Connection

## Really Weird Fact

$W(y_1, y_2)$ depends on ODE and almost not on $y_1$ and $y_2$!

$$y'' - 2y' + 2y = 0$$

Solutions: $ax + bx^2$