Solution

Solution of Cauchy–Euler

If m is a root of

$$m^2 + (a-1)m + b$$

then x^m satisfies

$$x^2 y'' + axy' + by = 0$$

- Note: $m \in \mathbb{R}$. *Not* $\mathbb{Z}$*!* $\mathbb{C}$
- 2 distinct real roots $\Longrightarrow$ 2 distinct solutions
- Potential problems at $x = 0$ (as expected)
- 1 real root or 2 complex roots not much harder
  But not on syllabus. . .

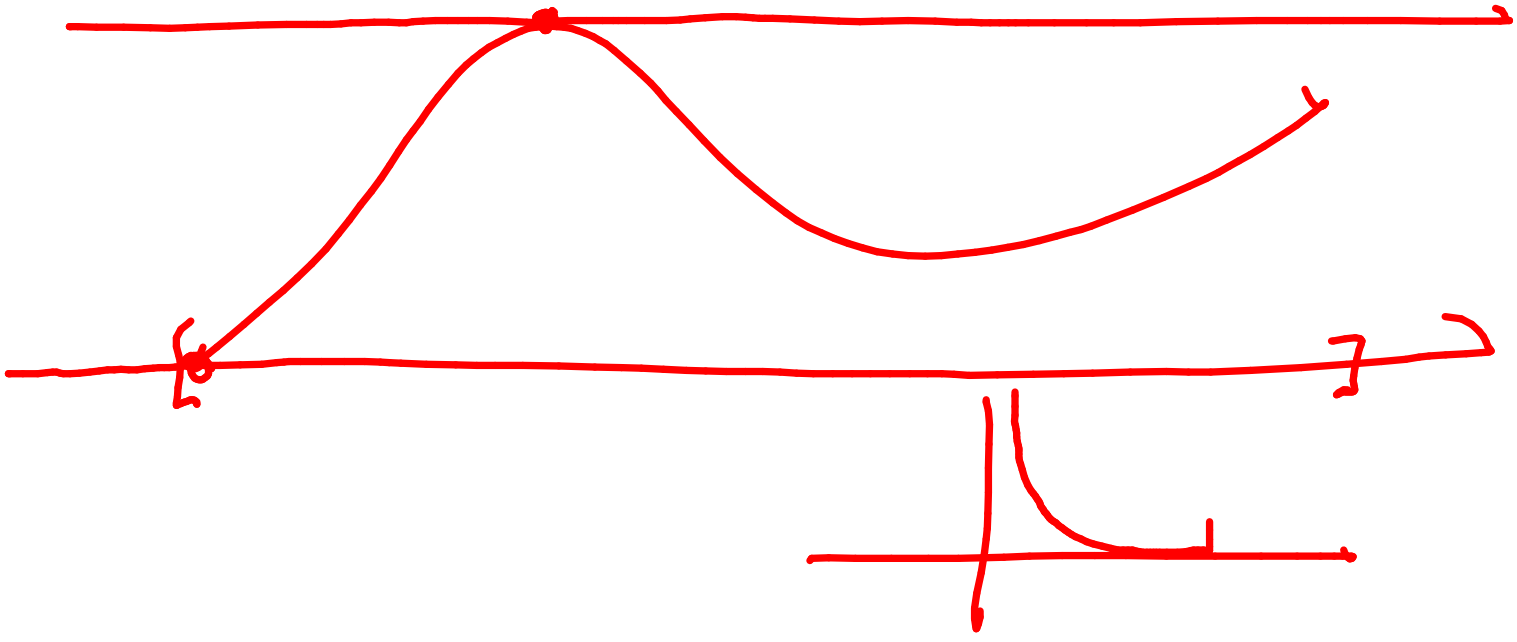
Uniqueness By Squeezing

Key Property of Continuous Functions

$f : [a, b] \to \mathbb{R}$ continuous

$\exists t_0 \in [a, b] : |f(t_0)| \geq |f(t)| \forall t \in [a, b]$

i.e. $|f|$ achieves its maximum



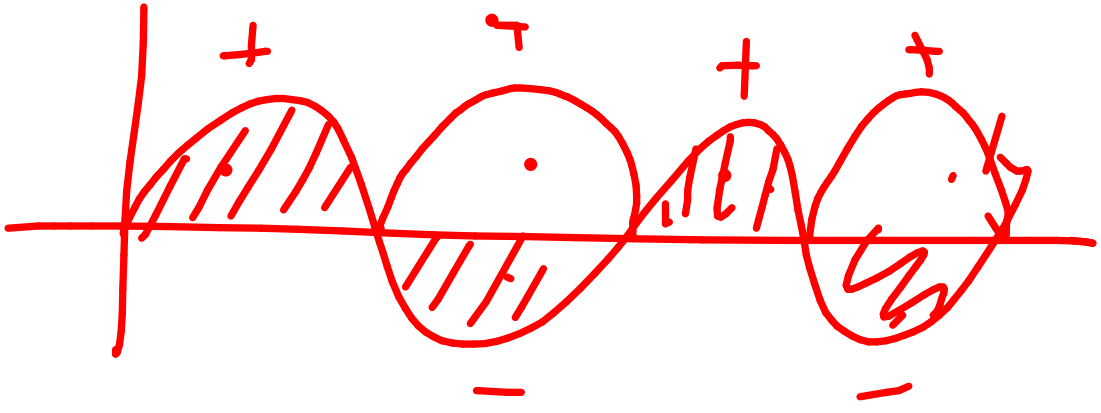
Uniqueness By Squeezing

Then:

$$y'(x_0) = \int_0^{x_0} y''(x)\mathrm{d}x$$

$$= -\int_0^{x_0} p(x)y'(x) + q(x)y(x)\mathrm{d}x$$

$$|y'(x_0)| \leq \int_0^{x_0} |p(x)||y'(x)| + |q(x)||y(x)|\mathrm{d}x$$



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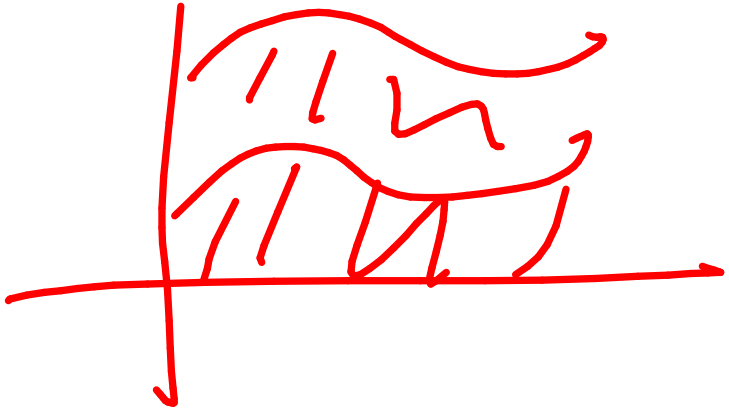
$$|y'(x_0)| \leq \int_0^{x_0} |p(x)||y'(x)| + |q(x)||y(x)|\mathrm{d}x$$

$$\leq \frac{1}{2}\int_0^{x_0} |y'(x_0)|\mathrm{d}x$$



Uniqueness Facts

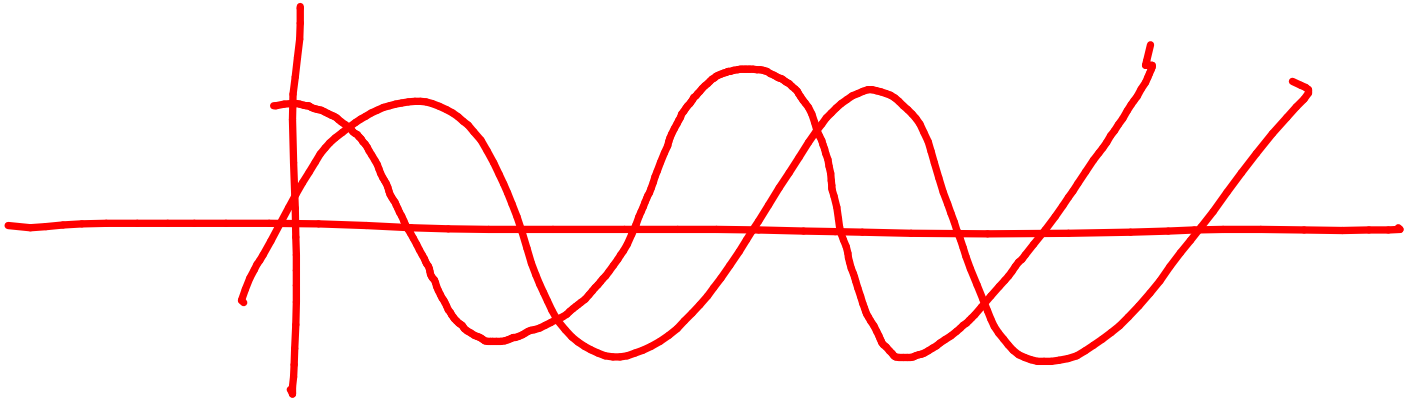
$$y'' + p(x)y' + q(x)y = 0$$

1. $y_1 \neq 0$ solution
   $\Longrightarrow y_1(x)$ and $y_1'(x)$ never simultaneously zero
   If so, y_1 solution with $y(x_0) = 0, y'(x_0) = 0$
   Uniqueness $\Longrightarrow y_1 = 0$
2. $y_1 \neq 0$, y_2 solutions
   $y_1(x_0) = 0 = y_2(x_0)$ some x_0
   $\Longrightarrow y_2 = k y_1$ some $k \in \mathbb{R}$
   Non-zero $\Longrightarrow y_1'(x_0) \neq 0$
   $\Longrightarrow -\frac{y_2'(x_0)}{y_1'(x_0)} y_1 + y_2$ satisfies 1

Uniqueness Example

Example

$$y'' + 4y = 0$$

Solutions: $\sin(2x)$, $\cos(2x)$, $\sin(x)\cos(x)$.

1. $\sin(2x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\sin(2x)$ never simultaneously zero