How to solve an ODE

Goal

- Find one or more functions that make the question true.
  - IVP $\Rightarrow$ one
  - not IVP $\Rightarrow$ two (order of ODE)

Step

1/ Can you write down a solution?

No?

1½/ Is it an IVP?

Yes: Find general solution & fit to conditions.

2/ Is it in standard form?

No? Put it in standard form. Start again.

3/ Classify:

i) Is it homogeneous?

In homogeneous

- Solve homogeneous
  - ↳ y_1, y_2

- Guess
  - based on ODE
    y_1, y_2
    RHS
  - Test candidates from List

Variation of Parameters

$$\left\{\begin{array}{l} y = c_1 y_1 + c_2 y_2 \\ y' = c_1 y_1' + c_2 y_2' \end{array}\right\} \quad c_1, c_2 \text{ functions}$$

$$\boxed{c_1' y_1 + c_2' y_2 = 0}$$

Homogeneous

Continue Classification

- Have I seen it before?

Yes! - Is it "constant coefficients"?

Yes: exponential solutions

- Is it "Euler-Cauchy"?

$$y'' + \frac{b}{x}y' + \frac{c}{x^2}y = 0$$

Yes: monomial

(not seen before)

- Guess one solution

Test candidates

- exponentials
  trigonometry
  polynomials
  constant
  logarithm

Use Reduction of Order

- $y = u y_1$
- Substitute in
- Solve for $U = u'$
- Integrate for u
- Multiply for y

Example (cont 2009,3)

$$y'' - \frac{3}{x}y' - \frac{5}{x^2}y = x^3$$

not IVP, is std form, not homogeneous

$\Rightarrow$ solve homogeneous

$$\boxed{y'' - \frac{3}{x}y' - \frac{5}{x^2}y = 0}$$

$$y'' - \frac{3}{x}y' - \frac{5}{x^2}y = 0$$

Euler-Cauchy

$$(x^2y'' - 3xy' - 5y = 0)$$

Try x^m:

$$m(m-1)x^{m-2} - \frac{3}{x}mx^{m-1} - \frac{5}{x^2}x^m = 0$$

$$m(m-1) - 3m - 5 = 0$$

$$m(m-1) - 3m - \overline{5} = 5$$

$$m^2 - 4m - \overline{5} = 0$$

$$m = -1 \text{ or } \overline{5}$$

$$x^{-1} \text{ and } x^{\overline{5}}$$

(Test!)

$$y = cx^{-1} + dx^5$$

$$y' = -cx^{-2} + 5dx^4$$

$$\Rightarrow \quad c'x^{-1} + d'x^5 = 0$$

$$y'' = -c'x^{-2} + 2cx^{-3} + 5d'x^4 + 20dx^3$$

$$-\frac{3}{x}\,y' = 3cx^{-3} - 15dx^3$$

$$-\frac{5}{x^2}\,y = -5cx^{-3} - 5dx^3$$

$$-c'x^{-2} + 5d'x^4 = x^3$$

$$c'x^{-1} + d'x^5 = 0 \quad (1)$$

$$-c'x^{-2} + 5d'x^4 = x^3 \quad (2)$$

$$\Rightarrow \quad d'x^5 + 5d'x^5 = x^4 \qquad (1) + x(2)$$

$$6d' = x^{-1}$$

$$d' = \frac{1}{6}x^{-1} \qquad d = \frac{1}{6}\log x$$

$$\Rightarrow \quad y = cx^{-1} + dx^5$$

$$c'x^{-1} + d'x^5 = 0 \quad (1)$$

$$-c'x^{-2} + 5d'x^4 = x^3 \quad (2)$$

$5(1) - x(2)$

$$\Rightarrow \quad 5c'x^{-1} + c'x^{-1} = x^4$$

$$6c'x^{-1} = x^4$$

$$c' = \frac{1}{6}x^5 \Rightarrow c = \frac{1}{36}x^6$$

Particular solution:

$$y = cx^{-1} + dx^5 = \frac{1}{36}x^5 + \frac{1}{6}(\log x)x^5 = \frac{1}{36}x^5(1 + 6\log x)$$

General solution: $\frac{1}{36}x^5(1 + 6\log x) + ax^{-1} + bx^5$

N.B. Can simplify to $\frac{1}{6}x^5 \log x + ax^{-1} + bx^5$ since b is arbitrary

Test: $y'' - \frac{3}{x}y' - \frac{5}{x^2}y = x^3$

$$y = \frac{1}{6}x^5 \log x$$

$$y' = \frac{5}{6}x^4 \log x + \frac{1}{6}x^4$$

$$y'' = \frac{20}{6}x^3 \log x + \frac{5}{6}x^3 + \frac{4}{6}x^3$$

$$\rightsquigarrow \quad \frac{20}{6}x^3 \log x + \frac{3}{2}x^3 - \frac{5}{2}x^3 \log x - \frac{1}{2}x^3 - \frac{5}{6}x^3 \log x$$

$$= x^3 \log x \left(\frac{20}{6} - \frac{15}{6} - \frac{5}{6}\right) + x^3\left(\frac{3}{2} - \frac{1}{2}\right) = x^3$$