TMA4115 Matematikk 3

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Introduction to Linear Systems

Today

1. Basic terminology
2. Visualisation of 2×2–system
3. Method of elimination and back substitution for 2×2– and 3×3–systems

Basic Terminology

$$ax + by = c$$

1 linear equation in 2 unknowns, x, y.
The solutions of this form a line in the x–y–plane (see figure 1).

$$\begin{aligned} x^3 + xy + \quad y &= 3 \\ x^2 + y^2 + \sqrt{z} &= 1 \end{aligned}$$

are **not** linear equations.

Definition 1. *A **linear system** is a family of linear equations in some variables (= unknowns).*

2 equations in 2 unknowns can be written

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned} \tag{1}$$

a solution of (1) is a pair of values (real numbers) that satisfy **both** equations.

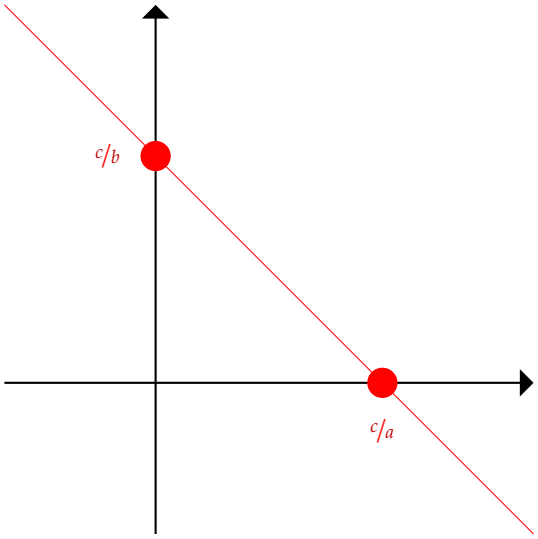

Figure 1: Solution set of $ax + by = c$

Examples

1.
$$\begin{aligned} 2x - \ \ y &= 5 \\ x + 2y &= 0 \end{aligned}$$

Solution: $x = 2$ and $y = -1$ since

$$\begin{aligned} 2(2) - \ (-1) &= 5 \quad \checkmark \\ (2) + 2(-1) &= 0 \quad \checkmark \end{aligned}$$

Not solution: $x = 3$ and $y = 1$ since

$$\begin{aligned} 2(3) - \ (1) &= 5 \quad \checkmark \\ (3) + 2(1) &= 5 \neq 0 \quad \times \end{aligned}$$

Definition 2. *A linear system is* ***consistent*** *if it has at least one solution. It is* ***inconsistent*** *if it has no solutions.*

Visualisation of 2×2–system

At this point, there was a computer demonstration

We observed: a 2×2–system, like (1), has a solution set of one of the following types:

1. Exactly one solution (2 non-parallel lines intersect)

2. No solution at all (2 parallel lines, not coincident)
3. An infinite family of solutions (2 (parallel) lines coincide)

Never: exactly, say, 6 solutions.

Method of Elimination

To determine the set of solutions.
2 unknowns:

- Add an appropriate multiple of the first equation to the second so that x disappears:
  - Determines y
  - Back substitution in first equation gives x

Examples

1.

$$\begin{array}{rl} 5x+3y=1 & \xrightarrow{R_1\leftrightarrow R_2} \quad x-2y=8 \\ x-2y=8 & \qquad\quad 5x+3y=1 \end{array}$$

$$\begin{array}{rl} & \xrightarrow{R_2-5R_1} \quad x-2y=8 \\ & \qquad\quad +13y=-39 \end{array}$$

$$\Longrightarrow y=-3$$

Back substitution in the first:

$$x=8+2y=8+2(-3)=2$$

2.

$$\begin{array}{rl} 2x+6y=4 & \xrightarrow{R_1\to 1/2R_1} \quad x+3y=2 \\ 3x+9y=11 & \qquad\quad 3x+9y=11 \end{array}$$

$$\begin{array}{rl} & \xrightarrow{R_2-3R_1} \quad x+3y=2 \\ & \qquad\quad 0=5 \end{array}$$

No solution.

3.

$$\begin{array}{rl} 2x+6y=4 & \xrightarrow{R_1\to 1/2R_1} \quad x+3y=2 \\ 3x+9y=6 & \qquad\quad 3x+9y=6 \end{array}$$

$$\begin{array}{rl} & \xrightarrow{R_2-3R_1} \quad x+3y=2 \\ & \qquad\quad 0=0 \end{array}$$

In this case, the second equation is trivially fulfilled so the set of solutions is the set of solutions of the first equation.

$$y = t \in \mathbb{R}, \quad x = 2 - 3y = 2 - 3t, \quad (x, y) = (2 - 3t, t), \forall t \in \mathbb{R}$$

4. Systems of 3 unknowns: if first equation involves x then add adequate multiple of first equation to second and third to eliminate x. The new second and third equations are a system with 2 unknowns, say y and z, apply to them the algorithm for 2 unknowns and finally substitute back in the first to find x.

$$\begin{array}{rl} x + 2y + z = 4 & \xrightarrow[R_2 - 3R_1]{R_3 - 2R_1} \quad x + 2y + z = 4 \\ 3x + 8y + 7z = 20 & \qquad\qquad\quad 2y + 4z = 8 \\ 2x + 7y + 9z = 23 & \qquad\qquad\quad 3y + 7z = 15 \end{array}$$

$$\begin{array}{rl} & x + 2y + z = 4 \\ \xrightarrow{R_2 \to 1/2 R_2} & y + 2z = 4 \\ & 3y + 7z = 15 \end{array}$$

$$\begin{array}{rl} & x + 2y + z = 4 \\ \xrightarrow{R_3 - 3R_2} & y + 2z = 4 \\ & z = 3 \end{array}$$

$$z = 3, \; y = 4 - 6 = -2, \; x = 4 + 4 - 3 = 5$$