Lecture 27: Revision: Linear Algebra

Meta-Theorem

Almost every linear algebra problem[a] can be solved using Gaussian Elimination.

[a] At least as far as **Matte 3** is concerned.

The general strategy for solving a linear algebra problem in **Matte 3** is the following:

1. Find the matrix.
2. Run Gaussian Elimination on that matrix.
3. Interpret the result.

Find the Matrix

Sometimes, it's obvious:

Exam Question: Kont 2011, Problem 4

Let

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$$

a) Find a basis for the null space, $\mathrm{Nul}(A)$, and a basis for the row space, $\mathrm{Row}(A)$.

b) Find a basis for the column space, $\mathrm{Col}(A)$. What is $\mathrm{Rank}(A)$?

The give-away is the line "Let $A = \ldots$".

However, sometimes it is not the obvious one, but one constructed from it.

Exam Question: Autumn 2011, Problem 4

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & -3 & 4 \\ 0 & 2 & 5 & -2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & -3 & 4 & 1 \\ 0 & 2 & 5 & -2 & a \end{bmatrix}$$

a) Find a basis for the solution space $\mathrm{Nul}(A)$ and a basis for the column space $\mathrm{Col}(A)$. Find the rank of A, $\mathrm{Rank}(A)$.

b) For which values of a is $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix}$ in $\mathrm{Col}(A)$?

c) Let T be a linear transformation with standard matrix A. Mark each of the following statements true or false (the answers should be justified).

1) T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$,

2) T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$,

3) T is onto,

4) T is one-to-one.

Part b) suggests that the matrix to use is actually the augmented matrix.

Sometimes, it is hidden. It might be a linear system:

Exam Question: Autumn 2010, Problem 5

Let $V \subseteq \mathbb{R}^4$ be the solution space of the system of equations

$$\begin{aligned} x + y - z + w &= 0 \\ x + 2y - 2z + w &= 0 \end{aligned}$$

1) Find an orthogonal basis for V.
2) Find the orthogonal projection of $\vec{b} = (1, 1, 1, 1)$ onto V.
3) Find an orthogonal basis for $\mathbb{R}^4$ where the first two vectors of this basis are the vectors you found in part a).

Here, the matrix has to be extracted from the linear system. In this case it is:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -2 & 1 \end{bmatrix}$$

The matrix might be given as a linear transformation:

Exam Question: Sprint 2012, Problem 5

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y + z \\ -x + 3y + z \\ 2x - z \\ y + 4z \end{bmatrix}$.

1) Find a matrix A such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

2) Find $\dim \mathrm{Nul}(A)$ and a basis for $\mathrm{Col}(A)$. Is T one-to-one (injective)? Is T onto (surjective)?

Part a) asks us explicitly to find the matrix of T. To do this, we extract the coefficients from the description. Thus

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

The matrix might be hidden as a family of vectors.

Exam Question: Kont 2010, Problem 6

For which values of the parameter a are the vectors $\vec{v}_1 = (1, -3, a)$, $\vec{v}_2 = (0, 1, a)$, and $\vec{v}_3 = (a, 2, 0)$ linearly independent?

In this case, we put the vectors together to create the matrix (being sure to use **column** vectors):

$$\begin{bmatrix} 1 & 0 & a \\ -3 & 1 & 2 \\ a & a & 0 \end{bmatrix}$$

Run Gaussian Elimination

1. Work from top-left to bottom-right,

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

2. Once we have processed $i-1$ rows and $j-1$ columns, we look at the remaining part of the matrix and pick a non-zero entry in the left-most non-zero column to be the next pivot.

$$\left[\begin{array}{c|cccc} * & * & * & * & * \\ \hline 0 & 0 & 0 & * & * \\ 0 & 0 & 2 & * & * \\ 0 & 0 & 4 & * & * \end{array}\right]$$

3. We swap this into the ith row.

$$\begin{bmatrix} * & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 2 & * & * \\ 0 & 0 & 4 & * & * \end{bmatrix} \qquad \begin{bmatrix} * & * & * & * & * \\ 0 & 0 & 2 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 4 & * & * \end{bmatrix}$$

4. Then use it to "purge" the lower non-zero entries from that column.

$$\begin{bmatrix} * & * & * & * & * \\ 0 & 0 & 2 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 4 & * & * \end{bmatrix} \times(-2) \begin{bmatrix} * & * & * & * & * \\ 0 & 0 & 2 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$$

5. Now we mark that row and column as "done" (and any columns to its left) and continue until we reach the edge of the matrix.

The result is in **row echelon form**:

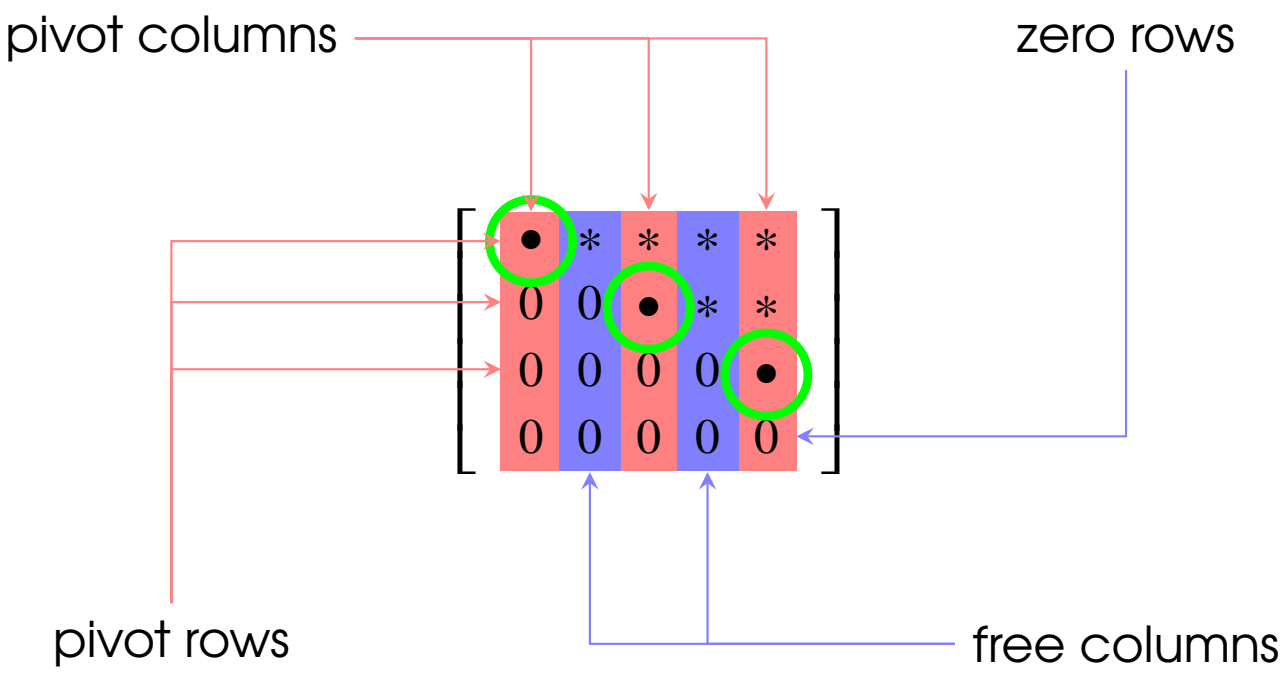
$$\begin{bmatrix} \bullet & * & * & * & * \\ 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots

reduced form:

$$\begin{bmatrix} 1 & * & 0 & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Interpret the Result



The exact interpretation depends on the question asked. The most important distinction is as to whether the matrix is the **coefficient** matrix or the **augmented** matrix.

The Coefficient Matrix

Pivot Columns

- Number of pivot columns = $\dim \mathrm{Col}(A)$ = rank of matrix
- Pivot columns in original matrix are a basis for the column space
- Every column a pivot column $\Longrightarrow$ matrix is injective, columns are linearly independent
- If the matrix is square, every column a pivot column $\Longrightarrow$ matrix is invertible

Free Columns

- Number of free columns = $\dim \mathrm{Nul}(A)$ = nullity of matrix
- Each free column gives a basis vector for the null space by setting that entry to 1 and the entries corresponding to the other free columns to 0, and setting the other entries accordingly
- There are free columns $\Longrightarrow$ matrix is not injective, columns are linearly dependent

Pivot Rows

- Number of pivot rows = $\dim \mathrm{Row}(A)$ = rank of matrix
- Pivot rows in echelon form are a basis for the row space
- Every row a pivot row $\Longrightarrow$ matrix is surjective, rows are linarly independent

Zero Rows

- There are zero rows $\Longrightarrow$ matrix is not surjective

The Augmented Matrix

When considering the augmented matrix we are interested in the existence and uniqueness of solutions to the equation $A\vec{x} = \vec{b}$.

Free Columns

- The last column is a free column $\Longleftrightarrow$ there is at least one solution.
- There are no other free columns $\Longleftrightarrow$ there is at most one solution.

Other Interpretations

- Gaussian elimination is also useful for calculating determinants:

$$\det A = (-1)^{\#\text{row swaps}} \prod \text{diagonal entries in echelon form}$$

$\times$ any row scale factors

- For a square matrix we can use the number of pivot columns to test for invertibility. If we determine that it is invertible, we can further use any of the equivalent characterisations of invertible matrices.

For an $n \times n$–**square** matrix A, the following are equivalent to the statement that A is invertible.

1. For any n–vector $\vec{b}$ the matrix equation $A\vec{x} = \vec{b}$ has a solution.
2. The linear transformation $\vec{x} \mapsto A\vec{x}$ is onto.
3. The columns of A span $\mathbb{R}^n$.
4. There is a $n \times n$–matrix C such that $AC = I_n$.
5. For any n–vector $\vec{b}$ the matrix equation $A\vec{x} = \vec{b}$ has at most one solution.
6. The matrix equation $A\vec{x} = \vec{0}$ has a unique solution.
7. The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.
8. The columns of A are linearly independent.
9. There is a $n \times n$–matrix D such that $DA = I_n$.
10. Every column is a pivot column.
11. Its reduced echelon form is I_n.
12. $\det A \neq 0$.
13. All of the above with A replaced by $A^\top$.

Possible Themes

1/ Finding bases 20

2/ Inverses 40

3/ Determinants 20

4/ Theory Lots

Autumn 2014, Problem 7

$Au = u$ eigenvectors

$Av = 0$ Null space & eigenvector

<u>Exect:</u> $w \neq 0$ st $Aw = \lambda w$

but if $\lambda = 0$, this means $Aw = 0w = 0$

We have eigenvalues $0, 1$ of A.

Assume $x = u + v$ works, try to find u & v.

Apply A: $Ax = A(u+v) = Au + Av$

$= u + 0 = u$

So if this works, we <u>must</u> have $u = Ax$

then we must have $v = x - u = x - Ax$.

If u & v exist, they must be $u = Ax$, $v = x - Ax$

and clearly, $x = Ax + (x - Ax)$

The last part is to show that u & v have the properties:

$$Au = A(Ax) = \underbrace{A^2x = Ax}_{\text{by assumption on } A} = u$$

$$Av = A(x - Ax) = Ax - A^2x = \underbrace{Ax - Ax}_{\text{by assumption on } A} = 0$$

Autumn 2012, 4

Find matrix of T: $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

$$\left[\begin{array}{cccccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

So the inverse matrix is $\begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$

So $T^{-1}(x_1, x_2, x_3) = \left(-\frac{9}{2}x_1 + 7x_2 - \frac{3}{2}x_3,\ -2x_1 + 4x_2 - x_3,\ \frac{3}{2}x_1 - 2x_2 + \frac{1}{2}x_3\right)$

Kout 2011, Problem 7 (I think)

1) $A^2 = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad 0 \cdot k + k \cdot 0$

$I + A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ has inverse $\frac{1}{\det(I+A)} \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$

$(I+A)^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$

2) $B^2 = 0$

Note that $(I+A)^{-1} = I - A$

Suggests we try $I - B$ for the inverse to $I + B$.

Multiply: $(I+B)(I-B) = I^2 + BI - IB - B^2$

$= I + B - B - B^2$

$= I - B^2$

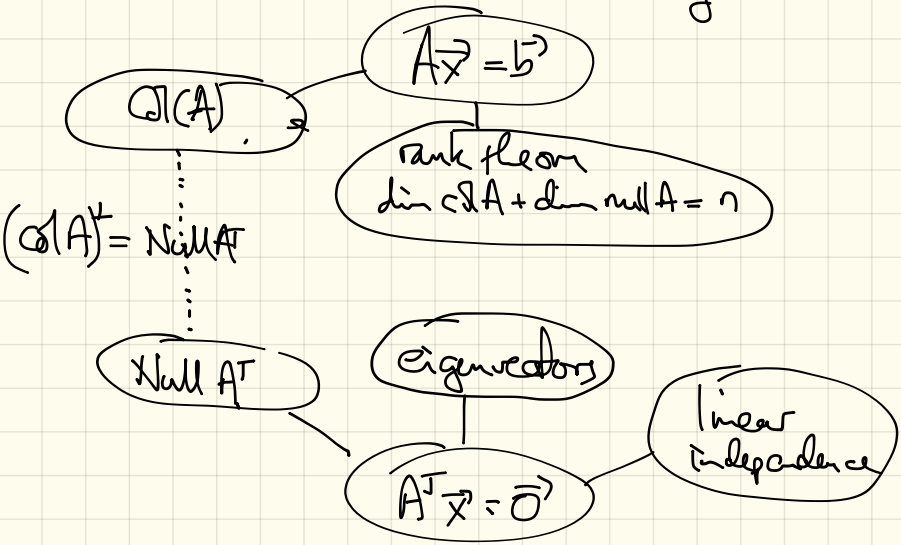
$= I$ by assumption on B

hence $I - B$ is the inverse to $I + B$

Spring 11, 8 (Maybe)

Suppose $A\vec{x} = \vec{b}$ always has a solution.

Show that $A^T\vec{x} = \vec{0}$ has only trivial solution.



$(\mathrm{Col}\,A)^\perp = \mathrm{Null}\,A^T$

We're told that $\mathrm{Col}\,A = \mathbb{R}^m$

so $\mathrm{Null}\,A^T = (\mathbb{R}^m)^\perp = \{\vec{0}\}$

thus $A^T\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$.