Math 3
Fall 2014
Solution to the exam.

Problem 1:

a) $$\frac{z_1}{z_2} = \frac{1/2\,(1+i\sqrt{3})}{1/2\,(\sqrt{2}+i\sqrt{2})} = \frac{(1+i\sqrt{3})(\sqrt{2}-i\sqrt{2})}{(\sqrt{2}+i\sqrt{2})(\sqrt{2}-i\sqrt{2})}$$

$$= \frac{\sqrt{2}+\sqrt{6} + i(\sqrt{6}-\sqrt{2})}{4}$$

$$= \frac{1}{4}(\sqrt{2}+\sqrt{6}) + \frac{1}{4}(\sqrt{6}-\sqrt{2})\, i$$

b) $z_1 = e^{i\pi/3}$; $z_2 = e^{i\pi/4}$

c) $$\frac{z_1}{z_2} = \frac{e^{i\pi/3}}{e^{i\pi/4}} = e^{i\pi/3} e^{-i\pi/4}$$

$$= e^{i(\pi/3 - \pi/4)} = e^{i\pi/12}.$$

d) $$\frac{z_1}{z_2} = e^{i\pi/12} = \cos\frac{\pi}{12} + i \sin\frac{\pi}{12}.$$

Besides, $$\frac{z_1}{z_2} = \frac{1}{4}(\sqrt{2}+\sqrt{6}) + \frac{1}{4}(\sqrt{6}-\sqrt{2})\, i$$

so $$\cos \pi/12 = \frac{1}{4}(\sqrt{6}+\sqrt{2})$$

$$\sin \pi/12 = \frac{1}{4}(\sqrt{6}-\sqrt{2}).$$

Problem 2 :

a) (E_h): $y'' - 4y' + 4y = g(x)$

$$\lambda^2 - 4\lambda + 4 = 0$$

$\Leftrightarrow$ $(\lambda - 2)^2 = 0$

so $\lambda = 2$ is the only root.

A fundamental system of solutions is

$$y_1 = e^{2x} \quad , \quad y_2 = x e^{2x} .$$

The general solution is

$$y = (c_1 + c_2 x) e^{2x} .$$

b) when $g(x) = e^{-2x}$, we look for

$$y_{p_1} = c\, e^{-2x}$$

$$4ce^{-2x} + 8ce^{-2x} + 4ce^{-2x} = e^{-2x}$$

so $16c = 1$ or $c = 1/16$

so $y_{p_1} = \frac{1}{16} e^{-2x}$

When $g(x) = e^{2x}$, we look for

$y_{p_2} = c\, e^{2x}$ $\rightarrow$ won't work because it is sol. of the homogeneous equation

then $y_{p_2} = c x e^{2x}$ $\rightarrow$ won't work for the same reason.

then $y_{p_2} = c x^2 e^{2x}$

$$y_{p_2}' = 2cx^2 e^{2x} + 2cx e^{2x}$$

$$y_{p_2}'' = 4cx^2 e^{2x} + 8cx e^{2x} + 2c e^{2x} .$$

Then

$$y_{p_2}'' - 4y_{p_2}' + 4y_{p_2} = e^{2x}$$

$$\Leftrightarrow \quad 2ce^{2x} = e^{2x} \quad \text{so} \quad c = \frac{1}{2}$$

$$\text{so} \quad y_{p_2} = \frac{1}{2}x^2 e^{2x}$$

c) By linearity, a particular solution for this equation is

$$y_p = \frac{1}{4}(y_{p_1} + y_{p_2}) = \frac{1}{64}e^{-2x} + \frac{1}{8}x^2 e^{2x}$$

so the general solution is

$$y = \frac{1}{64}e^{-2x} + \frac{1}{8}x^2 e^{2x} + (c_1 + c_2 x)e^{2x}.$$

Problem 3:

a- We can, for example, compute det A.

$$\det A = \begin{vmatrix} 2 & 4 \\ 3 & a \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & a \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

$$= 2a - 12 - a + 6 + 4 - 4$$

$$= a - 6$$

$$\det A \neq 0 \Leftrightarrow a \neq 6$$

so A is invertible if and only if $a \neq 6$.

b) assume $a \neq 6$, compute A^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & a & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & a-2 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & a-6 & 1 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/a-6 & -2/a-6 & 1/a-6 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/a-6 & -2/a-6 & 1/a-6 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1-(2/a-6) & 1+(4/a-6) & -2/a-6 \\ 0 & 0 & 1 & 1/a-6 & -2/a-6 & 1/a-6 \end{array}\right)$$

$$\text{so } A^{-1} = \frac{1}{a-6} \begin{pmatrix} 2a-12 & -a+6 & 0 \\ 4-a & a-2 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

Problem 4 :

a- $\det(A - \lambda I) = (\cdots) = -\lambda(\lambda-3)^2$

so the eigenvalues are 0 and 3.

b) for 0 : $A - 0I = A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

an eigenvector is $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

for 3 : $A - 3I = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

so eigenvectors are $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

c- $\vec{v}_2$ and $\vec{v}_3$ are independant, and are eigenvectors.
$\vec{v}_1$ is an eigenvector for a different eigenvalue.
so by theorem, $(\vec{v}_1, \vec{v}_3, \vec{v}_3)$ are independant.

We have three independant vectors in $\mathbb{R}^3$: that is a basis.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$ is a basis of $\mathbb{R}^3$, made of eigenvectors of A.

d) First, find an orthogonal basis.

because A is symmetric (or we can check it directly), $\vec{v_1}$ is orthogonal to $\vec{v_2}$ and $\vec{v_3}$.

Let's make $\vec{v_2}$, $\vec{v_3}$ ~~independant~~ orthogonal

let $\vec{u_2} = \vec{v_2}$

$$\cancel{\vec{u_3} = \vec{v_3} - \frac{\vec{u_2} \cdot \vec{v_3}}{\vec{v_2} \cdot \vec{v_2}}}$$

$$\vec{u_3} = \vec{v_3} - \frac{\vec{u_2} \cdot \vec{v_3}}{\vec{u_2} \cdot \vec{u_2}} \vec{u_2}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

Now, let's make them orthonormal:

$$\|\vec{v_1}\| = \sqrt{1^2+1^2+1^2} = \sqrt{3} \quad ; \quad \|\vec{u_2}\| = \sqrt{2}$$

$$\|\vec{u_3}\| = \frac{\sqrt{3}}{\sqrt{2}}.$$

so let $\vec{w_1} = \dfrac{\vec{v_1}}{\|\vec{v_1}\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$, $\vec{w_2} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$, $\vec{w_3} = \begin{pmatrix} -\sqrt{6}/6 \\ -\sqrt{6}/6 \\ \sqrt{6}/3 \end{pmatrix}$

now, $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an ON basis of $\mathbb{R}^3$ made of eigenvectors

e) So $\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{D} = P^T A P$.

with $P = \begin{pmatrix} \sqrt{3}/3 & -\sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & 0 & \sqrt{6}/3 \end{pmatrix}$,

and P is orthogonal ($P^T = P^{-1}$).

Problem 5:

a) The system is $A\vec{x} = \vec{b}$ with

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$

b) Write the augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{array}\right)$$

There is a row "$0 = 1$" so the system has no solution.

c) The ~~solution~~ least-square solution of $A\vec{x} = \vec{b}$ is the solution of

$$A^T A \vec{x} = A^T \vec{b}$$

$$A^T A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 23 \\ 10 \end{pmatrix}$$

so we solve:

$$\begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}^{-1} = \frac{1}{42-36} \begin{pmatrix} 3 & -6 \\ -6 & 14 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 3 & -6 \\ -6 & 14 \end{pmatrix}$$

so $\vec{x} = (A^T A)^{-1} (A^T \vec{b})$

$$= \frac{1}{6} \begin{pmatrix} 3 & -6 \\ -6 & 14 \end{pmatrix} \begin{pmatrix} 23 \\ 10 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 69 - 60 \\ 140 - 138 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 9 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/3 \end{pmatrix}$$

so $\vec{x} = \begin{pmatrix} 3/2 \\ 1/3 \end{pmatrix} \cancel{\begin{pmatrix} 9/8 \\ 1/4 \end{pmatrix}}$ is the least square solution.

Problem 6 :

a- Write the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 8 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1/2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right) \quad \text{so } \vec{x} = \begin{pmatrix} 2 \\ -1/2 \\ 1/2 \end{pmatrix} \text{ or } \begin{array}{l} x_1 = 2 \\ x_2 = -1/2 \\ x_3 = 1/2 \end{array}$$

b) The matrix A is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

c) Question a) showed that A has three pivots. So A is invertible.

d) Using question a), we find $\vec{x} = \begin{pmatrix} 2 \\ -1/2 \\ 1/2 \end{pmatrix}$

e) $p_{\vec{x}}(4) = 2 + \left(-\frac{1}{2}\right) \times 4 + \frac{1}{2} \times 4^2 = 2 - 2 + 8 = 8.$

Problem 7

Case 1 : if either $\vec{u}$ or $\vec{v}$ are orthogonal to $\vec{w}$, (for example $\vec{u}$ is orthogonal to $\vec{w}$), then

$\vec{u} = 1\vec{u} + 0\vec{v}$ answers the question.

(it is not 0, a LC of $\vec{u}$ and $\vec{v}$, and is orthog. to $\vec{w}$).

Case 2. We assume neither $\vec{u}$ or $\vec{w}$ are orthogonal to $\vec{w}$, so $\vec{u}\cdot\vec{w} \neq 0$ and $\vec{v}\cdot\vec{w} \neq 0$.

We look for a and b such that

$a\vec{u} + b\vec{v}$ is orthogonal to $\vec{w}$

so $(a\vec{u} + b\vec{v})\cdot\vec{w} = 0$

so $(\vec{u}\cdot\vec{w})\,a + (\vec{v}\cdot\vec{w})\,b = 0$

We can take for example

$$a = 1 \text{ and } b = -\frac{\vec{u}\cdot\vec{w}}{\vec{v}\cdot\vec{w}}$$

which exists because the denominator is not 0.

so: $\vec{u} + \left(-\frac{\vec{u}\cdot\vec{w}}{\vec{v}\cdot\vec{w}}\right)\vec{v}$ is

- a linear combination of $\vec{u}$ and $\vec{v}$
- orthogonal to $\vec{w}$
- not zero because $(\vec{u}, \vec{v})$ are independant