Examination paper for **TMA4110 Matematikk 3**

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**Examination date:** December 4\(^{th}\), 2014

**Examination time (from–to):** 09:00–13:00

**Permitted examination support material:** C: Simple Calculator (Casio fx-82ES PLUS, Citizen SR-270X, Citizen SR-270X College, or Hewlett Packard HP30S), Rottmann: Matematiske formelsamling

**Other information:**
Give reasons for all answers, ensuring that it is clear how the answer has been reached. Each exercise has the same weight.

**Language:** English

**Number of pages:** 3

**Number pages enclosed:** 0

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**Checked by:**

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Date Signature
Problem 1  In this exercise, we consider the complex numbers
\[ z_1 = \frac{1}{2} + i \frac{\sqrt{3}}{2} \quad \text{and} \quad z_2 = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}. \]

a) Write \( z_1/z_2 \) in the form \( z_1/z_2 = a + ib \) (do not use the \( \cos \) or \( \sin \) functions).

b) Compute the modulus and an argument of \( z_1 \) and \( z_2 \). Write \( z_1 \) and \( z_2 \) in polar form.

c) Write \( z_1/z_2 \) in the form \( z_1/z_2 = \rho e^{i\theta} \).

d) Deduce from the above the values of \( \cos(\pi/12) \) and \( \sin(\pi/12) \).

Problem 2  In this exercise, we consider the differential equation
\[ y'' - 4y' + 4y = g(x). \]

a) Compute the general solution of the homogeneous equation.

b) Compute a particular solution when \( g(x) = e^{-2x} \) and when \( g(x) = e^{2x} \).

c) Compute the general solution of the equation when
\[ g(x) = \frac{1}{4}(e^{-2x} + e^{2x}). \]

Problem 3  Consider the matrix
\[ A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{bmatrix}. \]

a) For which values of \( a \) is this matrix invertible?

b) Compute \( A^{-1} \), when this inverse exists.
Problem 4  
In this exercise, we consider the matrix $A$ given by

$$
A = \begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix}.
$$

a) Find the eigenvalues of $A$?

b) Find a non-zero eigenvector for each eigenvalue of $A$.

c) Find a basis of $\mathbb{R}^3$ made of eigenvectors of $A$.

d) Find an orthonormal basis of $\mathbb{R}^3$ made of eigenvectors of $A$.

e) Find an orthogonal matrix $P$ and a diagonal matrix $D$ such that $D = P^T A P$.

Problem 5

a) Given the data pairs

\[
\begin{align*}
a_1 &= 1, \quad b_1 = 2, \\
a_2 &= 2, \quad b_2 = 3, \\
a_3 &= 3, \quad b_3 = 5,
\end{align*}
\]

express the system

\[
\begin{align*}
a_1 x_1 + x_2 &= b_1 \\
a_2 x_1 + x_2 &= b_2 \\
a_3 x_1 + x_2 &= b_3
\end{align*}
\]

of linear equations in matrix form $A\mathbf{x} = \mathbf{b}$: What are $A$, $\mathbf{x}$ and $\mathbf{b}$?

b) For $A$ and $\mathbf{b}$ as in (b), show that $A\mathbf{x} = \mathbf{b}$ does not have a solution.

c) Use the least squares method to find an approximate solution $\mathbf{x}$ for the equation $A\mathbf{x} = \mathbf{b}$.

d) For $\mathbf{x}$ as in (d), sketch the three data points and the line $b = x_1 a + x_2$ into a coordinate system.

e) For $\mathbf{x}$ as in (d), compute $4x_1 + x_2$?
Problem 6

a) Solve the following system of linear equations:

\[ \begin{align*}
    x_1 + x_2 + x_3 &= 2 \\
    x_1 + 2x_2 + 4x_3 &= 3 \\
    x_1 + 3x_2 + 9x_3 &= 5.
\end{align*} \]

b) Let

\[ p_x(t) = x_1 + x_2 t + x_3 t^2 \]

denote the polynomial with real coefficients \( x_1, x_2, x_3 \in \mathbb{R} \). The transformation

\[ \mathbb{R}^3 \rightarrow \mathbb{R}^3 \]

\[ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} p_x(1) \\ p_x(2) \\ p_x(3) \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + 2x_2 + 4x_3 \\ x_1 + 3x_2 + 9x_3 \end{bmatrix} \]

is linear. Find the matrix \( A \) that describes this linear transformation.

c) For \( A \) as in (b), show that \( A \) is invertible.

d) For \( A \) as in (b), find \( \mathbf{x} \) such that

\[ A \mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}. \]

e) For \( \mathbf{x} \) as in (d), compute \( p_x(4) = x_1 + 4x_2 + 16x_3 \).

Problem 7

Let \( \mathbf{u} \) and \( \mathbf{v} \) be two nonzero, independent vectors in \( \mathbb{R}^3 \). Let \( \mathbf{w} \) be a nonzero vector in \( \mathbb{R}^3 \). Show that there exists a non-zero linear combination of \( \mathbf{u} \) and \( \mathbf{v} \) which is orthogonal to \( \mathbf{w} \).