Oppgaver Tirsdag 15. August

1 Finn e^{At} for

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ b) $A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ c) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

d) $A = \dfrac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$ e) $A = \dfrac{1}{2}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ f) $A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$

2 Løs initialverdiproblemene

a) $\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ b) $\mathbf{x}' = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ d) $\mathbf{x}' = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

Løsning Tirsdag 15. August

1 a) Vi husker fra oppgavene på fredag at

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

er diagonaliserbar som

$$A = PDP^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Dermed

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^0 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

b)

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

er diagonaliserbar som

$$A = PDP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

Dermed

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 + e^t & -1 + e^t \\ -1 + e^t & 1 + e^t \end{bmatrix} \end{aligned}$$

c) Vi gjenkjenner A til å være en Jordan block, så vi vet at

$$A = 2I_2 + N = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Nå, siden $2I_2$ og N kommuterer ($2I_2N = 2N = N2 = NI_22 = N(2I_2)$), har vi

$$e^{At} = e^{2I_2t}e^{Nt}.$$

Dermed,

$$e^{2I_2t} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} = e^{2t}I_2$$

og

$$\begin{aligned} e^{Nt} &= I_2 + \frac{t}{1!}N + \frac{t^2}{2!}N^2 + \frac{t^3}{3!}N^3 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{t}{1!}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{t}{1!}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \end{aligned}$$

gir oss

$$e^{At} = e^{2t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

d) Vi begynner med å finne egenverdiene til A.

$$\rho_A(\lambda) = \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \lambda \end{vmatrix} = (\frac{1}{\sqrt{2} - \lambda})^2 + \frac{1}{2}$$
$$= \frac{1}{2} - \frac{2}{\sqrt{2}} + \lambda^2 + \frac{1}{2}$$
$$= \lambda^2 - \sqrt{2}\lambda + 1$$

$$\lambda = \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$$

Vi har komplekse egenverdier, så vi faktoriserer A som PCP^{-1}, hvor

$$P = \begin{bmatrix} \mathrm{Re}\,\mathbf{v} & \mathrm{Im}\,\mathbf{v} \end{bmatrix}$$

for en egenvektor $\mathbf{v}$ tilhørende egenverdien $\lambda = a - b = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$ av A, og

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Den observante leser har muligens allerede sett at $A = C$, og at vi dermed kunne droppet å finne egenvektorer. Vi gjennomfører likevel hele prosessen.
Egenvektoren til $\lambda = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$ finner vi som

$$\mathrm{Null}\left(\begin{bmatrix} i\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \end{bmatrix}\right) = \mathrm{Null}\left(\begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right)$$

Altså er $\begin{bmatrix} 1 \\ i \end{bmatrix}$ egenvektoren vi ser etter, og siden $\mathrm{Re}\,\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ og $\mathrm{Im}\,\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, får vi

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = P^{-1}.$$

Dermed

$$A = PCP^{-1} = ICI = C$$

som den observante leser så litt lenger oppe.
Vi skriver nå A som

$$A = \frac{1}{\sqrt{2}}I_2 + \frac{1}{\sqrt{2}}S, \text{ for } S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

og får

$$e^{At} = e^{t/\sqrt{2}}e^{1/\sqrt{2}\ S} = e^{t/\sqrt{2}} \begin{bmatrix} \cos(t/\sqrt{2}) & -\sin(t/\sqrt{2}) \\ \sin(t/\sqrt{2}) & \cos(t/\sqrt{2}) \end{bmatrix}$$

e)

$$\begin{vmatrix} 2-\lambda & -1 \\ 1/2 & 1/2-\lambda \end{vmatrix} = (2-\lambda)(1/2-\lambda)+1$$
$$= 1 - 2\lambda - 1/2\lambda + \lambda^2 + 1/2$$
$$= \lambda^2 - 2.5\lambda + 1.5$$

$$\lambda = \frac{2.5 \pm \sqrt{25/4 - 6}}{2} = \frac{5/2 \pm \sqrt{1/4}}{2}\frac{5 \pm 1}{4} = \begin{cases} 3/2 \\ 1 \end{cases}$$

Vi finner egenvektorer til $\lambda = 3/2$:

$$\begin{bmatrix} 1/2 & -1 \\ 1/2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Vi finner egenvektorer til $\lambda = 1$:

$$\begin{bmatrix} 1 & -1 \\ 1/2 & -1/2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Dermed er A diagonaliserbar som

$$A = PDP^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Dette gir oss da

$$e^{At} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{1.5t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2e^{1.5t} & e^t \\ e^{1.5t} & e^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2e^{1.5t} - e^t & -2e^{1.5t} + 2e^t \\ e^{1.5t} - e^t & -e^{1.5t} + 2e^t \end{bmatrix}$$

f) $A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$ kan diagonaliseres som

$$A = PDP^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \frac{1}{-8} \begin{bmatrix} -2 & -1 \\ -2 & 3 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}.$$

Dermed

$$e^{At} = \frac{1}{8} \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}$$
$$= \frac{1}{8} \begin{bmatrix} 3e^{3t} & e^{-5t} \\ 2e^{3t} & -2e^{-5t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}$$
$$= \frac{1}{8} \begin{bmatrix} 6e^{3t} + 2e^{-5t} & 3e^{3t} - 3e^{-5t} \\ 4e^{3t} - 4e^{-5t} & 2e^{3t} + 6e^{-5t} \end{bmatrix}$$

2

a) $\mathbf{x} = e^{At} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3e^t \\ 2 \end{bmatrix}$

b) $\mathbf{x} = e^{At} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c) $\mathbf{x} = e^{At} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = e^{2t} \begin{bmatrix} 1 + 4t \\ 4 \end{bmatrix}$

d) $\mathbf{x} = e^{At} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4e^{t/\sqrt{2}} \begin{bmatrix} \cos(t/\sqrt{2}) - \sin(t/\sqrt{2}) \\ \cos(t/\sqrt{2}) + \sin(t/\sqrt{2}) \end{bmatrix}$