05.09 - Plenumsregning 1

02 September 2022 15:09

Tema: Komplekse tall

Oppgave 1 - Ekstraoppgaver

1. Beregn og merk av i det komplekse planet. Husk at det kan være lurt å bruke polar form.

$$z = r(\cos\theta + i\sin\theta) \qquad \underline{z = re^{i\theta}}$$

b) $\underbrace{(1+i)}_{z_1}\cdot\underbrace{(1+\sqrt{3}i)}_{z_2}$

Plan: Skrive z som et produkt av $z_1 = r_1e^{i\theta_1}$ og $z_2 = r_2e^{i\theta_2}$

$$z_1 \rightarrow r_1e^{i\theta_1}$$

$$r_1 = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \underline{\sqrt{2}}$$

$$\theta_1 = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

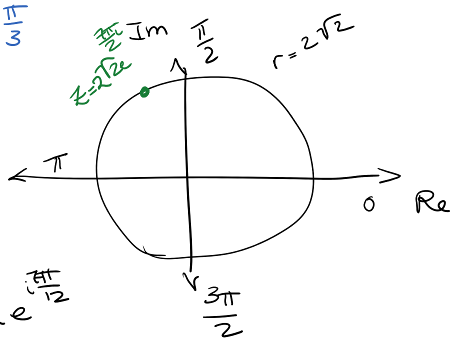
$$\underline{z_1 = \sqrt{2}\,e^{\frac{\pi}{4}i}}$$

$$z_2 \rightarrow r_2e^{i\theta_2}$$

$$r_2 = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = \underline{2}$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$z_2 = 2e^{\frac{\pi}{3}i}$$



$$z = z_1\cdot z_2 = \sqrt{2}e^{\frac{\pi}{4}i}\cdot 2e^{\frac{\pi}{3}i} = 2\sqrt{2}\,e^{\frac{\pi}{4}i+\frac{\pi}{3}i} = \underline{2\sqrt{2}\,e^{i\frac{7\pi}{12}}}$$

Oppgave 2 - Ekstraoppgaver

2. Løs ligningene

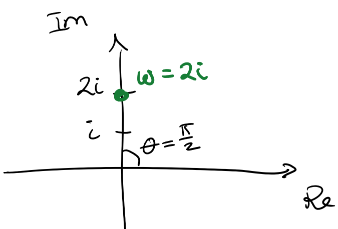
b) $z^3 = 2i$

OPPSKRIFT

1. Skriv z på polarform: $z = re^{i(\theta+2\pi k)}$
2. Røttene er
$$z_k = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n}+\frac{2\pi}{n}k\right)}$$
3. Det er n ulike komplekse tall. For $k = 0, 1, \dots, n-1$ regner vi ut
$$\sqrt[n]{r}\,e^{i\left(\frac{\theta}{n}+\frac{2\pi}{n}k\right)}$$

Løsning

$$z^3 = 2i$$



1. La $\omega = 2i$

$$r = \sqrt{a^2+b^2} = \sqrt{0^2+2^2} = \underline{2}$$

2. $$z_k = \sqrt[n]{r}e^{i\left(\frac{\theta}{n}+\frac{2\pi}{n}k\right)} = \sqrt[3]{r}\,e^{i\left(\frac{\theta}{3}+\frac{2\pi}{3}k\right)}$$

3. $$z_1 = \sqrt[3]{2}\,e^{i\left(\frac{\pi/2}{3}+\frac{2\pi}{3}\cdot 0\right)} = \underline{\sqrt[3]{2}\,e^{\frac{\pi}{6}i}}$$

$$z_2 = \sqrt[3]{2}\,e^{i\left(\frac{\pi/2}{3}+\frac{2\pi}{3}\cdot 1\right)} = \underline{\sqrt[3]{2}\,e^{\frac{5\pi}{6}i}}$$

$$z_3 = \sqrt[3]{2}\,e^{i\left(\frac{\pi/2}{3}+\frac{2\pi}{3}2\right)} = \underline{\sqrt[3]{2}\,e^{\frac{9\pi}{6}i}}$$

Løsninger av likningen $z^3 = 2i$

Oppgave 3 - Ekstraoppgaver

3. La $z = a + bi$. Finn real- og imaginærdelen til

Den konjugerte av nevneren:

$$\overline{(a^2-b^2)+2abi} = (a^2-b^2) - 2abi$$

d) $\frac{1}{z^2}$

$$\frac{1}{z^2} = \frac{1}{(a+bi)^2} = \frac{1}{a^2+2abi+\overset{=-1}{i^2}b^2} = \frac{1}{(a^2-b^2)+2abi}$$

$$= \frac{1\,\big((a^2-b^2)-2abi\big)}{\big((a^2-b^2)+2abi\big)\big((a^2-b^2)-2abi\big)}$$

Bruker 3. □-setning: $(x+y)(x-y) = x^2-y^2$

$$= \frac{a^2-b^2-2abi}{(a^2-b^2)^2-(2abi)^2}$$

$$= \frac{a^2-b^2-2abi}{(a^2-b^2)^2-(-2^2a^2b^2)}$$

$$= \frac{a^2-b^2-2abi}{(a^2-b^2)^2+(2ab)^2}$$

$$= \underbrace{\frac{a^2-b^2}{(a^2-b^2)^2+(2ab)^2}}_{\text{Reelle del}} + i\underbrace{\frac{-2ab}{(a^2-b^2)^2+(2ab)^2}}_{\text{Imaginær del}}$$

Oppgave 4 - Ekstraoppgaver

4. La $z = re^{i\theta}$, og gjenta oppgaven over.

Mål: Skrive z på formen $a+ib$. Plan: Polarform.

$$z = r(\cos\theta + i\sin\theta) = \underbrace{r\cos\theta}_{a} + \underbrace{r\sin\theta}_{b}\, i$$

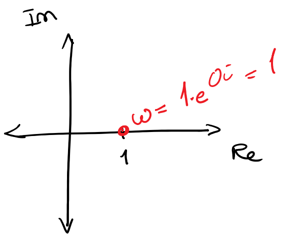
Oppgave 8 - Ekstraoppgaver

8. Noen artige polygoner.

a) Finn alle tredjerøttene til 1. Tegn en rett linje fra løsning til løsning, etter økende vinkel. Hva slags geometrisk figur er dette?

$$w = \overset{a}{1} + \overset{b}{0}i = 1$$

① Vil skrive w på formen $w = re^{i\theta}$



$r = 1$

$\theta = 0$

② Tredjerøttene er $\sqrt[3]{r}\,e^{i\left(\frac{\theta}{3}+\frac{2\pi}{3}k\right)}$

$k = 0, 1, 2$

③ $k = 0$:

$$w_0 = \sqrt[3]{1}\,e^{i\left(\frac{0}{3}+\frac{2\pi}{3}\cdot 0\right)} = 1$$

$k = 1$:

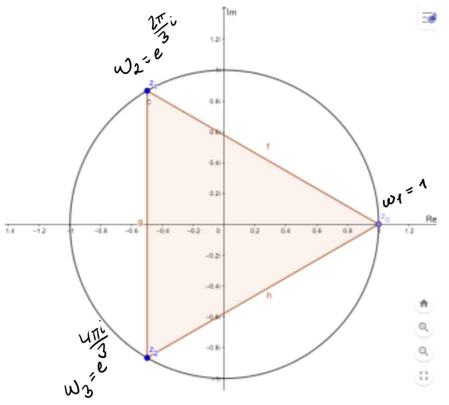
$$w_1 = \sqrt[3]{1}\,e^{i\left(\frac{0}{3}+\frac{2\pi}{3}\cdot 1\right)} = e^{\frac{2\pi}{3}i}$$

$k = 2$:

$$\sqrt[3]{1}\,e^{i\left(\frac{0}{3}+\frac{2\pi}{3}\cdot 2\right)} \quad e^{\frac{4\pi i}{3}}$$

$\underline{k=2:}$

$$\omega_2 = \sqrt[3]{1}\, e^{i\left(\frac{0}{3}+\frac{2\pi}{3}\cdot 2\right)} = \underline{\underline{e^{\frac{4\pi i}{3}}}}$$



Oppgave 9 - Ekstraoppgaver

9. La $z \neq 0$ og $w \neq 0$ være komplekse tall. Vis at $zw \neq 0$.

Vi vet: $z, w \in \mathbb{C}$, $z \neq 0$, $w \neq 0$

Vi vil vise: $z \cdot w \neq 0$

Idé: Skrive på polarform

$\underline{\text{Bevis:}}$

$z = re^{i\theta}$ $\qquad r \neq 0$

$w = pe^{i\phi}$ $\qquad w \neq 0$

$$z \cdot w = re^{i\theta} \cdot pe^{i\phi} = \underbrace{r \cdot p}_{\neq 0} \cdot \underbrace{e^{i\theta + i\phi}}_{\neq 0} \neq 0$$

$\underline{\text{Altså må } z \cdot w \neq 0}$ $\quad \square$

Eksamen høst 2018 Oppgave 1

a) Skriv det komplekse tallet z = −1 + i √ 3 på polar form.

b) Vis at z = −1 + i √ 3 er en sjetterot av 64.

c) Skisser z^6 = 64 i det komplekse planet.

Rektangulær form $\qquad$ Polarform

a) $z = \underset{a}{-1} + \underset{b}{\sqrt{3}}\,i \longrightarrow z = r(\cos\theta + i\sin\theta)$

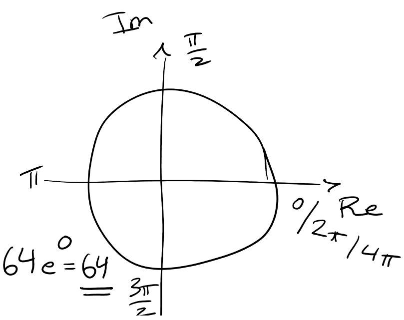
$r = |z| = \sqrt{a^2+b^2} = \sqrt{(-1)^2+(\sqrt{3})^2} = \underline{2}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right) + \pi \cdot k = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) + \pi \cdot 1 = \underline{\frac{2\pi}{3}}$

$$\underline{z = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)}$$

b) $z = re^{i\theta} = \underline{2e^{i\frac{2\pi}{3}}}$

$z^6 = \left(2e^{i\frac{2\pi}{3}}\right)^6 = 2^6 \cdot \left(e^{i\frac{2\pi}{3}}\right)^6 = 64 \cdot e^{i\frac{2\pi}{3}\cdot 6} = 64e^{4\pi i} = 64e^0 = \underline{\underline{64}}$



b) $z = r e^{\ldots} = 2e^{\ldots}$

$$z^6 = \left(2e^{i\frac{2\pi}{3}}\right)^6 = 2^6 \cdot \left(e^{i\frac{2\pi}{3}}\right)^6 = 64 \cdot e^{\frac{2\pi}{3}\cdot 6} = 64e^{4\pi i} = 64e^0 = \underline{\underline{64}}$$

$\frac{3\pi}{2}$ $12\pi / 4\pi$

c) 6-te røttene $\Rightarrow$ 6 løsn.

Skissere alle z s.a. $z^6 = 64$ i det komplekse planet

$$z_k = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)}$$

$r = 64 \qquad \theta = 0$

<u>$k = 0$:</u>

$$z_0 = \sqrt[6]{64}\, e^{i\left(\cancel{\frac{0}{6}}^{0} + \cancel{\frac{2\pi}{6}\cdot 0}^{0}\right)} = \underline{2}$$

<u>$k = 1$:</u>

$$z_1 = \sqrt[6]{64}\, e^{i\left(\cancel{\frac{0}{6}}^{0} + \frac{2\pi}{6}\cdot 1\right)} = \underline{2e^{\frac{\pi}{3}i}}$$

$\underline{z_2 = 2e^{\frac{2\pi i}{3}}}$

$\underline{z_3 = 2e^{\pi i}}$

$\underline{z_4 = 2e^{\frac{4\pi}{3}i}}$

$\underline{z_5 = 2e^{\frac{5\pi}{3}i}}$