An Introductory Course to Linear Algebra and Differential Equations

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Based on the notes of Olav Ersland and Kristoffer Varholm

Contents

1 Preliminaries

1.1 Sets

We begin by introducing some important definitions.

Definition 1.1 (Sets)**.** A **set** is a collection of unordered, distinct elements. We usually denote a set by a capital letter, though there are always exceptions. Two sets are the same if all their elements are identical.

This definition is very unrestrictive. A set can comprise of anything, as we see in the following examples.

Example 1.2. The following are all sets

- A set of numbers, $A = \{1, 2, 3\}$,

- a set of shapes, $B = \{\square, \triangle, \circ\}$,
- a set of names, $C = \{\text{Steve}, \text{Everest}\}$,
- even a set of sets, $\{A, B, C\}$.

Always remember, each object must be distinct. There is no repetition of objects in sets, and there certainly no concept of order required by a set. For instance, $\{1, 2, 3\} = \{3, 1, 2\} = \{2, 1, 3\}$.

There is no restriction on size for a set. If a set A contains a finite number number of elements, we denote its **cardinality** $|A|$ to be the number of elements in A. Though there could be a non finite number of elements, and for many of the sets we will be considering this is the case!

Example 1.3 (Some important sets)**.** We now introduce some important sets.

- The set of **natural numbers**, $\mathbb{N} = \{1, 2, 3, \dots\}$. Sometimes 0 is included, other times people denote the set $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$.
- The set of **integers**, $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.
- The set of all **rational numbers**, $\mathbb{Q}$. This is the set of all numbers that can be written as $\frac{a}{b}$ where a and b are in $\mathbb{Z}$. i.e., the set of all fractions of integers.
- The set of **real numbers**, $\mathbb{R}$. This is any number on the line. It contains all the elements of the previous sets, and more. For instance, π is in $\mathbb{R}$, but not in the three previous sets, as it cannot be written as a fraction of integers. This is not an easy set to define, and we wont go into details, but just think of it as any decimal number.
- The **empty set**, $\emptyset$, is the set which contains no elements.

Interestingly, some sets containing a non-finite number of elements have a defined size. However, to say this with confidence requires a lot of work.

Importantly, we have two distinct categories. Countable, and uncountable sets. The real numbers $\mathbb{R}$ are uncountable. Every other set we have introduce so far is countable. Countable means that the set is finite, or every element can be related to an element of the natural numbers uniquely. This means there is a so called "bijection" between the set and the naturals. We will explore bijections later.

In these notes, we will mainly be working with the naturals, integers or the reals.

Definition 1.4 (Some common operations and notations)**.** Let X and Y be some sets.

- We say X is a **subset** of Y if every element of X is contained in Y. This is denoted $X \subset Y$.
- The **union** of two sets A and B, denoted $A \cup B$, is the set containing every element containing in A or B.
- The **intersection** of two set A and B, denoted $A \cap B$, is the set containing every element contained in both A and B.
- The symbol $\in$ can be read as "in". For instance, if we write $x \in X$, we mean that x is contained in X.

What does a subset look like? The following example should clear this definition up.

Example 1.5 (Some subsets)**.** A simple example of a subset would be $\{1, 2, 4\} \subset \{1, 2, 3, 4, 5\}$. Non finite sets can also be subsets of each other. We see from the sets above that $\emptyset \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. In fact, the empty set $\emptyset$ is a subset of any set.

Example 1.6 (Unions and Intersections)**.** Consider $A = \{1, 2, 4, 5, 7\}$ and $B = \{3, 5, 8, 9\}$. Then

- $A \cup B = \{1, 2, 3, 4, 5, 7, 8, 9\}$,
- $A \cap B = \{5\}$.

If two sets X and Y share no elements their intersection is the empty set $\emptyset$.

In some works you may see the use of **quantifiers**. We will try to avoid them.

- The symbol $\forall$ can be read as "for all", "for any", or "for every",
- the symbol $\exists$ can be read as "there exists",
- the rarely used symbol $\exists!$ can be read as "there exists only one".

1.2 Functions

Definition 1.7. A **function** is a relationship/process that associates each element of some set, called a domain, to **exactly one** element in another set, called the range.

A function f with domain X and range Y is denoted by $f : X \to Y$. The relationship is defined using the notation $x \mapsto f(x)$.

Note. Importantly, $f(x)$ is not a function, but an element of the set Y, though in some lazy circumstance it is easy to write $f(x) = \dots$ to write the function. In such a case, we must give a domain, and if no range is given we assume it is the largest possible range for the given domain.

We introduce some examples of functions. This very general definition allows for lots of freedom.

Example 1.8 (Some functions)**.** We list some functions.

- $f : \mathbb{R} \to \mathbb{R}, x \mapsto e^x$,
- A piecewise function, $H : \mathbb{R} \to \{0, 1\}$, called the Heaviside step function

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

- A function that maps even numbers to odd numbers, and odd numbers to even numbers, $g : \mathbb{N} \to \mathbb{N}$,

$$n \mapsto \begin{cases} n - 1, & n \text{ even}, \\ n + 1, & n \text{ odd}. \end{cases}$$

2 Linear Algebra in $\mathbb{R}^N$

2.1 Euclidean Vectors

We now introduce the most common vector, the ones used to model the real world. An abstract notion of a vector needs to be based off the properties that these vectors satisfy.

Definition 2.1 (Vector in $\mathbb{R}^n$)**.** Let N be some number in $\mathbb{N}$. Denote by $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R}$ the set of all n-tuples of real numbers. We write these in a few common notations. For $\mathbf{v} \in \mathbb{R}^n$,

- $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ (**column vector notation**),
- $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ (**row vector notation**),
- $\mathbf{v} = (v_1, v_2, \dots, v_n)$ (**coordinate notation**).

We wish to explore some properties of vectors. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then the following properties hold

- $\alpha\mathbf{v} = \alpha \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}$ (**scalar multiplication**),
- $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$ (**vector addition**).

We also have some geometric properties of vectors. Note that vectors have a sense of direction and length,

- the **dot product/inner product** of two vectors is given by $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_2 + u_2 v_2 + \dots u_n v_n$,
- The **magnitude/length** of a vector is given by $|v| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}$,
- the angle between two vectors, θ is given by the relation $\mathbf{u} \cdot \mathbf{v} = |u||v| \cos(\theta)$,
- two vectors are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$, or if the angle between them is a right angle.

2.2 Linear Transformations

Definition 2.2 (Linear Transformation)**.** We call a function $T : \mathbb{R}^n \to \mathbb{R}^m$ a **linear transformation** if it preserves scalar multiplication and vector addition, that is

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

for any $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$.

Example 2.3 (Some linear transformations)**.** The following are some examples of linear transformations:

- $\mathrm{Id} : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (**Identity Mapping**),

- $S : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$,
- $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x_1, x_2, x_3) = (x_1, 2x_2, x_3)$,
- $P : \mathbb{R}^3 \to \mathbb{R}^2, P(x_1, x_2, x_3) = (x_1, x_2)$.

You can confirm these are linear transformations directly using the definition.

We also introduce an examples of a mapping that is **not** a linear transformation.

- $\tilde{T} : \mathbb{R}^2 \to \mathbb{R}^2, (x_1, x_2) \mapsto (x_1^2, x_2)$.

Linear transformations are particularly nice functions, however in the forms given they are necessarily nice to work with. Is there a way we can write these in a nice way?

2.3 Matrices

Definition 2.4 (Matrix)**.** A **matrix** is a rectangular array of numbers. We denote the set of matrices with m rows and n columns $\mathbb{R}^{m\times n}$, and we write $A \in \mathbb{R}^{m\times n}$ as

$$A = [a_{i,j}] = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}.$$

Matrix addition and scalar multiplication are defined element wise, analogous to the definitions for vectors in Definition 2.1.

A matrix with 1 row, or 1 column, is a vector. A matrix with 1 row and 1 column is usually denoted as a scalar.

Multiplication of a vector $x \in \mathbb{R}^n$ by a matrix $A \in \mathbb{R}^{m\times n}$ is given by

$$Ax = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{bmatrix}$$

or more conveniently, it is a vector with entries

$$(Ax)_i = \sum_{j=1}^{n} a_{i,j}x_j.$$

Immediately we see a nice property of matrices. **Any mapping defined by a matrix is a linear transformation**. That is to say a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ given by

$$T(x) = Ax, \qquad (x \in \mathbb{R}^n) \tag{1}$$

with some given matrix $A \in \mathbb{R}^{m\times n}$ is always a linear transformation. You can check the properties required of a linear transformation as an exercise.

These linear transformation are nice to work with, as we will see later. In fact, things are even better than this, as we see in our first theorem.

Theorem 2.5. *let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$-matrix A such that*

$$T(x) = Ax, \qquad (x \in \mathbb{R}^n).$$

That is, any linear transformation can be written using the help of some matrix.

Example 2.6 (Matrices for some Linear Transformations)**.** Consider the linear transformations from Example 2.3. We will rewrite these using a matrix.

- $\mathrm{Id} : \mathbb{R}^2 \to \mathbb{R}^2$ has the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
- $S : \mathbb{R}^2 \to \mathbb{R}^2, S(x) = Ax$ has the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,
- $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x) = bx$ has the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Definition 2.7 (Matrix Multiplication)**.** The next important operation that we introduce is **matrix multiplication**. Given two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, we define

$$AB = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & \cdots & b_{1,p} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,p} \end{bmatrix} := \begin{bmatrix} a_{1,1}b_{1,1} + \ldots a_{1,n}b_{n,1} & \cdots & a_{1,1}b_{1,p} + \cdots + a_{1,n}b_{n,p} \\ \vdots & \ddots & \vdots \\ a_{m,1}b_{1,1} + \cdots + a_{m,n}b_{n,1} & \cdots & a_{m,1}b_{1,p} + \cdots + a_{m,n}b_{n,p} \end{bmatrix}.$$

In other word, AB is the matrix with entries

$$(AB)_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}, \qquad i = 1, \ldots, m, j = 1, \ldots, p.$$

Definition 2.8 (Composition)**.** Given two linear maps $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^p \to \mathbb{R}^n$ we define their **composition** $S \circ T : \mathbb{R}^p \to \mathbb{R}^m$ by

$$(S \circ T)(x) = S(T(x)).$$

If we S and T in Definition 2.8 are given by matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ respectively, then

$$(T \circ S)(x) = ABx.$$

2.4 Linear Systems of Equations

We now introduce the fundamental part of linear algebra, the thing the entire subject is built around. Analysing and solving linear systems of equations

Definition 2.9 (Linear Systems)**.** A **linear equation** is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where we know the coefficients $a_1, \ldots, a_n$ and b, and the $x_1, \ldots, x_n$ are the unknown variables.

A **linear system** is a collection of one or more linear equations,

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2, \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m. \end{aligned}$$

Suppose the coefficients and the b_i are real numbers. We can then rewrite this as

$$Ax = b,$$

where A is an $m \times n$-matrix with entries $a_{i,j}$, $i = 1, \ldots, n, j = 1, \ldots, m$ and b is an m-vector with entries $b_i, i = 1, \ldots, m$. We are looking for a solution $x \in \mathbb{R}^n$.

The trick to solving these systems is knowing the operations (manipulations) that we can perform on the equations that preserve the solution. There are three operations

- Interchanging two equations,
- Multiplying an equation by a non-zero constant,

- Adding equations to one another.

To simplify things, we rewrite the system in the form of a matrix, where we ignore the x vector. This is called an **augmented** matrix.

$$\left[\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_n \end{array}\right]$$

For each of the three operation introduced above, there is an equivalent operation that can be performed on this matrix

- Swapping two rows,
- Multiplying a row by a constant,
- Adding rows to each other.

We perform these operations until the matrix is in a form that allows us to easily extract the x which solve the system. The method of performing these operations is called **Gaussian Elimination**.

Definition 2.10 (Row Echelon Form)**.** A matrix is **row echelon form** if two conditions are satisfied.

- All rows containing only zeros are at the bottom,
- The leading coefficient of a non-zero row is strictly to the right of the above rows leading coefficient.

If the leading coefficient of all the non-zero rows is 1, then the matrix is said to be in **reduced row echelon form**.

It is not immediately obvious why this method works, and how to use it. To illustrate this, we have an example.

Example 2.11. We wish to solve the system

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 16 \\ 3x_1 + x_2 + 4x_3 &= 27 \\ 4x_1 + x_2 + x_3 &= 18. \end{aligned}$$

Re-writing this as an augmented matrix, we have

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 16 \\ 3 & 1 & 4 & 27 \\ 4 & 1 & 1 & 18 \end{array}\right].$$

Performing the row operations,

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 16 \\ 3 & 1 & 4 & 27 \\ 4 & 1 & 1 & 18 \end{array}\right] \xrightarrow{r_3-2r_1, 2r_2-3r_1} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 16 \\ 0 & -7 & 5 & 6 \\ 0 & -5 & -1 & -14 \end{array}\right]$$

$$\xrightarrow{7r_3-5r_2} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 16 \\ 0 & -7 & 5 & 6 \\ 0 & 0 & -32 & -128 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{32}r_3} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 16 \\ 0 & -7 & 5 & 6 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\xrightarrow{r_1-r_3, r_2-5r_3} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 12 \\ 0 & -7 & 0 & -14 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Rewriting this as a system, we have

$$2x_1 + 3x_2 + 0x_3 = 12$$

$$0x_1 - 7x_2 + 0x_3 = -14$$

$$0x_1 + 0x_2 + 1x_3 = 4.$$

Thus $x_3 = 4$, $x_2 = 2$. Plugging these into the first equation, we have $x_1 = 3$.

3 Vector Spaces over $\mathbb{R}$

The toolbox of linear algebra applies to a lot more than the vectors reflecting the real world. We are now leaving the space of $\mathbb{R}^n$, and looking at spaces which share its properties.

Definition 3.1 (Vector Space)**.** Let V be set with two operations, denoted scalar multiplication $(\cdot)$ and vector addition $(+)$, satisfying 8 axioms:

For all $u, v \in V$, and for all $\alpha, \beta \in \mathbb{R}$,

- $u + (v + w) = (u + v) + w$ (**associativity**),
- $u + v = v + u$ (**commutativity**),
- $\alpha(\beta v) = (\alpha\beta)v$ (**Compatibility**),
- $\alpha(u + v) = \alpha u + \alpha v$ (**distributivity (+)**)
- $(\alpha + \beta)u = \alpha u + \beta u$ (**distributivity $(\cdot)$**)
- There exists an element $\mathbf{0} \in V$ such that for all $u \in V$, $\mathbf{0} + u = u$, (**Identity element of addition**),
- For each $u \in V$ there exists and element $-u \in V$ such that $u + -u = 0$, (**Inverse elements of addition**),
- $1v = v$ (**identity element of scalar multiplication**).

Then V is a **vector space** (over the real numbers $\mathbb{R}$), and $v \in V$ is called its vectors.

Example 3.2 (Some examples of vector spaces)**.** The following are all vector spaces. You can check the axioms are satisfied as an exercise.

- As we copied these properties from $\mathbb{R}^n$, it is of course a vector space,
- the set of all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$, or equivalently the set of all $m \times n$-matrices,
- Polynomials p up to a given degree n ($\mathbb{P}_n$), or polynomials are arbitrary degree ($\mathbb{P}$),
- The set of all functions $f : \mathbb{R} \to \mathbb{R}$.

It can be annoying to check all the axioms of a vector space. Subspaces are an easy way to form another vector space, or to conclude a space is indeed is a vector space.

Definition 3.3 (Subspace)**.** Let V be a vector space. A set $U \subseteq V$ is a **subspace** if

- $\mathbf{0} \in U$,
- U is closed under the vector space operations of V, that is for any $u, v \in U$ and $\alpha \in \mathbb{R}$,

- $u + v \in U$,
- $\alpha u \in U$.

Example 3.4 (Subspaces)**.** The following are all examples of subspaces,

- $\{\mathbf{0}\}$ and V are always subspaces of V,
- $\{(\alpha, \beta, 0) \mid \alpha, \beta \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$,
- $\{(\alpha_1, \ldots, \alpha_n) \mid \alpha_i = 0\}$ for some $i = 1, \ldots, n$ is a subspace of $\mathbb{R}^n$,
- $\mathbb{P}_n$ is a subspace of $\mathbb{P}_m$ for $m > n$, which is a subset of $\mathbb{P}$, which is a further subset of all functions $f : \mathbb{R} \to \mathbb{R}$.

We will now introduce a few of the core definitions of linear algebra.

Definition 3.5 (Linear Combination and Linear Independence)**.** Given $n \in \mathbb{N}$ and $v_1, \ldots, v_n \in V$, a **linear combination** is an expression

$$\alpha_1 v_1 + \cdots + \alpha_n v_n$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

We say the vectors $v_1, \ldots, v_n$ are **linearly independent** if

$$\alpha_1 v_1 + \cdots + \alpha v_n = 0,$$

if an only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Opposite to this, the vectors $v_1, \ldots, v_n$ are **linearly dependent** if they are not linearly independent.

Linear independence is equivalent to saying "no vector in $\{v_1, \ldots v_n\}$ can be written as a linear combination of the other vectors", i.e. for each $k = 1, \ldots n$ there is no $\alpha_1, \ldots \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n \in \mathbb{R}$ such that

$$v_k = \alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1} + \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n.$$

Example 3.6 (Linear Independence)**.** How do we check if the vectors are linearly independent? Consider the following examples.

- $\begin{bmatrix}1\\3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\6\\2\end{bmatrix}$ are linearly independent. To check this, we solve for the possible $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \begin{bmatrix}1\\3\\1\end{bmatrix} + \beta \begin{bmatrix}1\\6\\2\end{bmatrix} = 0$$

  which is just a linear system.

- $\begin{bmatrix}1\\3\\1\end{bmatrix}$, $\begin{bmatrix}1\\6\\2\end{bmatrix}$, and $\begin{bmatrix}2\\9\\3\end{bmatrix}$ are linear dependent. This is easy to see, as

$$\begin{bmatrix}1\\3\\1\end{bmatrix} + \begin{bmatrix}1\\6\\2\end{bmatrix} = \begin{bmatrix}2\\9\\3\end{bmatrix},$$

  so the third vector is a linear combination of the other vectors.

Definition 3.7 (Span, Dimension, and Bases)**.** Let V be a vector space.

Given $v_1, \dots v_n \in V$, the **span** of these vectors is the set of all linear combinations. That is, it is

$$\operatorname{span}\{v_1, \dots, v_n\} \coloneqq \{\alpha_1 v_1 + \cdots + \alpha_n v_n \mid \alpha_1, \dots, \alpha_n \in \mathbb{R}\}.$$

The **dimension** of V, $\dim V$, is the maximum number of linearly independent vectors in V. That is to say, if there exists n vectors $v_1, \dots v_n \in V$ that are linearly independent, but $n+1$ vectors $v_1, \dots v_{n+1} \in V$ are always linearly dependent, then the dimension $\dim V = n$. The dimension of $\{0\}$ is 0.

A **basis** of V is a maximal (dimension n) set of linearly independent vectors. If a set $\{v_1, \dots, v_n\}$ is a basis of V, then any vector in V can be represented as a unique linear combination of this basis. i.e. for any $u \in V$ there exists a unique $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$u = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

Note. Given a basis $\{v_1, \dots, v_n\}$ of a vector space V of dimension n, we have that any vector u can be expressed a linear combination with unique coefficients $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. In other words, this linear combination can be represented using a vector $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$. This connection between

any vector space of finite dimension and the vectors in $\mathbb{R}^n$ is extremely powerful, and will play a role later on.

Example 3.8 (Some Bases)**.** We have a few examples of bases.

- The standard basis in $\mathbb{R}^n$ is the set of vectors $\{e_1, \ldots, e_n\}$. Here e_i $(i = 1, \ldots, n)$ is the matrix with entries
$$(e_i)_j = \begin{cases} 0 & j \neq i \\ 1 & j = i. \end{cases}$$
- The matrices $\mathbb{R}^{m\times n}$ have a basis $\{E_{i,j} \mid i = 1, \ldots, m, j = 1, \ldots, n\}$, where $E_{i,j}$ is the matrix with 1 as the i, j-th entry, and $0's$ elsewhere.
- $\mathbb{P}_n$, the polynomials of degree n, has the basis $\{1, x, \ldots, x^n\}$.

3.1 A Return to Matrices and Linear Transformations

Lets start looking at the some common spaces associated with matrices.

Definition 3.9 (Row Space, Column Space, and the Null Space)**.** Let $A \in \mathbb{R}^{m\times n}$. We matrix can be thought of a collection of column vectors or row vectors, written as

$$A = \begin{bmatrix} \boldsymbol{c_1} & \boldsymbol{c_2} & \cdots & \boldsymbol{c_n} \end{bmatrix} \text{ or } A = \begin{bmatrix} \boldsymbol{r_1} \\ \boldsymbol{r_2} \\ \vdots \\ \boldsymbol{r_m} \end{bmatrix}.$$

The **row/column space** is the vector space given by the span of the row/column vectors

$$\mathrm{Col}(A) \coloneqq \mathrm{span}\{\boldsymbol{c_1}, \cdots, \boldsymbol{c_n}\},$$

$$\mathrm{Row}(A) \coloneqq \mathrm{span}\{\boldsymbol{r_1}, \cdots, \boldsymbol{r_n}\}.$$

If A defines a linear transformation $T(x) = Ax$, then the column space is the image of T,

$$\mathrm{Col}(A) = \mathrm{Im}(T) \coloneqq T(\mathbb{R}^n) = \{T(x) \mid x \in \mathbb{R}^n\}.$$

The **null space** of A is the set of all vectors which A (or T) maps to zero

$$\mathrm{Null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} = \{x \in \mathbb{R}^n \mid T(x) = 0\}.$$

We now have a few important theorems concerning the dimensions of these spaces.

Theorem 3.10 (Rank)**.** *Let $A \in \mathbb{R}^{m\times n}$. The dimension of the column space is the same as the dimension of the row space. We call this the* ***rank*** *of the matrix A.*

$$\mathrm{Rank}(A) \coloneqq \dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A)).$$

Theorem 3.11 (Rank-Nullity)**.** *The* ***nullity*** *of a matrix $A \in \mathbb{R}^{m\times n}$ is the dimension of the null space. The following holds*

$$\mathrm{Rank}(A) + \mathrm{Nullity}(A) = n.$$

The question is, how do we find the bases for these two spaces? Notice that if one of the rows is a linear combination of the others, then after performing a sufficient sequence of elementary row operations that row will become the zero row. Hence elementary row operations preserve the basis of the row space. They also preserve the basis of the null space.

So, the trick to finding the basis for the row space is to perform Gaussian elimination until the matrix is in row echelon form. This reduced matrices rows will form the basis.

Lets look at this in action.

Example 3.12. The matrix

$$A = \begin{bmatrix} 2 & 6 & 14 \\ 1 & 4 & 11 \\ 0 & 3 & 12 \end{bmatrix}.$$

Reducing this with Gaussian elimination is

$$\tilde{A} = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

The basis for the row space is then

$$\left\{ \begin{bmatrix} 1 & 3 & 7 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 4 \end{bmatrix} \right\},$$

hence its dimension is 2.

Using the rank nullity theorem, we see that the dimension of the null space is 1. To find the null space's basis, we can solve $\tilde{A}x = \mathbf{0}$ for x. Writing this as a system we see

$$\begin{aligned} x_1 + 3x_2 + 7x_3 &= 0 \\ 0 + x_2 + 4x_3 &= 0 \\ 0x_3 &= 0. \end{aligned}$$

Solving this we find this system is satisfied for any choice of x_3. Let t be a free real number, and set $x_3 = t$. Then $x_2 = -4t$, and thus $x_1 = 5t$. Or

$$x = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} t,$$

satisfies $Ax = 0$ for any real number $t \in \mathbb{R}$. So a basis for the null space is

$$\left\{ \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

The trick for the column space is to look at the pivots of the reduced matrix (the columns with a 1 at the bottom) and the respective columns in the original matrix for a basis for the column space. So for this example the basis is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix} \right\}.$$

Lets return to looking at Linear systems. Let $A \in \mathbb{R}^{m \times n}$, and consider the system

$$Ax = b. \tag{2}$$

Let's start answering the natural questions that occur when we have a problem.

When does a solution to this system exist? Another way of writing the problem is

$$x_1 \mathbf{c_1} + \cdots + x_n \mathbf{c_n} = b.$$

where $c_1, \ldots, c_n$ are the column vectors for A. Hence we know the problem has a solution if there exists a linear combination of the column vectors equal to b. Or, more simply, b lies in the column space of A

$$b \in \mathrm{Col}(A).$$

When is the solution unique? It's best if there is only one solution to the problem.

Suppose that there exists two solutions, x and y in $\mathbb{R}^n$ with $x \neq y$. Then

$$Ax = b = Ay.$$

So

$$A(x-y) = Ax - Ay = b - b = 0,$$

or rather $x - y$ lies in the null space of A.

If $\text{Null}(A) = \{0\}$, then $x - y = 0$, or rather $x = y$, a contradiction. In other words, the solution is unique if the $\text{Null}(A) = \{0\}$.

On the other hand, suppose the solution is unique, Suppose also $\text{Null}(A)$ contains more than just 0. Then for $y \in \text{Null}(A)$ with $y \neq 0$,

$$A(x+y) = Ax + Ay = b + 0 = b,$$

or $x + y$ is another solution to the problem, a contradiction. So then we must have $\text{Null}(A) = \{0\}$. The null space in this situation has dimension zero, hence the $Rank(A) = n$ by the rank nullity theorem.

Summing up,

Lemma 3.13. *The solution to $Ax = b$ is unique if and only if the null space only contains the zero element, or equivalently the rank of A is n.*

Suppose we remove all the rows containing only zeros in A and b. If $m < n$ then the system never has a unique solution, as there are more equations than unknowns. This is called an overdetermined system.

3.2 Inverses

Let A be a square matrix, $A \in \mathbb{R}^{n \times n}$, for whom $rank(A) = n$. Then there exists a **matrix inverse**, a unique matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I,$$

where I is the $n \times n$ identity matrix (the matrix which has ones down the diagonal, and zeros elsewhere).

Under these assumptions, the solution to the problem $Ax = b$, for $b \in \mathbb{R}^n$, is unique.

Further,

$$Ax = b \iff A^{-1}Ax = A^{-1}B \iff Ix = A^{-1}b \iff x = A^{-1}b,$$

or in other words, the solution to the problem is $x = A^{-1}b$.

For functions, if f is an injective (one-to-one), and surjective (onto), then f is called bijective. In which case, there exists an inverse function f^{-1} satisfying

$$f \circ f^{-1} = \mathrm{Id} \text{ and } f^{-1} \circ f = \mathrm{Id}.$$

For a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, the inverse T^{-1} is linear if it exists.

How do we find the inverse matrix? We don't usually want to calculate the inverse matrix, as it can be time consuming, or computationally expensive on computers. However, if you wish to calculate the inverse of a matrix $A \in \mathbb{R}^{n\times n}$ then you can do so by performing Gaussian elimination on the matrix

$$\left[A \,\middle|\, I\right] = \left[\begin{array}{ccc|ccc} a_{1,1} & \cdots & a_{1,n} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} & 0 & \cdots & 1 \end{array}\right].$$

Do so until you reach a matrix of the form

$$\left[I \,\middle|\, B\right]$$

and the matrix B will be the inverse matrix.

3.3 Determinants

We are now going to explore a very useful mathematical object for the analysis and solving of system of equations given by square matrices. The determinant is a scalar value calculated using the entries of a matrix.

We start by defining the determinant for square matrices of size $1, 2$ and 3. Let $A \in \mathbb{R}^{n\times n}$. If

- $n = 1$: $\det(A) = \det(a_{1,1}) = a_{1,1}$,
- $n = 2$: Then
$$\det(A) = \det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$
- $n = 3$: Then
$$\begin{aligned} \det(A) = \det &\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \\ &= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\ &\quad - a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}. \end{aligned}$$

For the first two, the formula should be easy to remember. For the third, things are starting to get too complicated. Before we can begin talking about the calculation for general matrices, we need to introduce some notation.

Definition 3.14 (Minor)**.** For a square matrix $A \in \mathbb{R}^{n\times n}$, the $\mathbf{(i,j)^{th}}$**-minor** is the determinant of the matrix formed by deleting the ith row and j^{th} column. It is denoted by

$$M_{i,j} = \det \begin{bmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{bmatrix}.$$

Using this, we can define the determinant,

Definition 3.15 (Determinant)**.** Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. The **determinant** of a matrix A is a scalar value given by

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} M_{k,j} a_{k,j}.$$

where $k \in \{1, \ldots, n\}$ can be chosen arbitrarily.

Thus the determinant can be calculated via a recursive process. We know how to calculate it for matrices of size $1, 2$ and 3. The determinant of a matrix of size 4 is given by the sum of 3 minors, which are the determinants of matrices of size 3. Hence we can calculate the determinant of matrix of size 4. The same can be done for matrices of size 5, and so on.

The choice of k determines which row you go across doing this method. Always choose the row with the most zeros. This also works over a column.

Example 3.16. Lets calculate the determinant of the following matrix.

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 1 & 4 & 0 \\ 1 & 5 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}.$$

We go down the third column. The zero entries will not contribute to the determinant, so we ignore them. The other two entries we make crosses through

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 1 & 4 & 0 \\ 1 & 5 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 1 & 4 & 0 \\ 1 & 5 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}.$$

The determinants of the matrices ignoring these rows and columns are the minors we will use when calculating the determinant.

$$M_{2,3} = \det \begin{bmatrix} 1 & 3 & 2 \\ 1 & 5 & 1 \\ 3 & 0 & 2 \end{bmatrix} = -17,$$

and

$$M_{4,3} = \det \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix} = 13.$$

Combining things together, we have

$$\det(A) = (-1)^5 \cdot (-17) \cdot 4 + (-1)^7 \cdot 13 \cdot 1 = 55.$$

Geometrically, the absolute value of the determinant of a 2×2-matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

is the area of the parallelepiped spanned by the matrices column or row vectors. The area of this matrix will be zero if only if

$$0 = |\det(A)| = |ad - bc|$$

or also

$$ad = bc.$$

This will only be the case if the following two vectors are parallel,

$$\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}.$$

In other words, the rows/columns of the matrix must be linearly dependent (as parallel means one is a multiple of the other). On the other hand, the determinant is non-zero if the columns/rows are linearly independent, i.e. the matrix has full rank, 2.

We can summarise a lot in one handy Lemma.

Lemma 3.17. *Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. The following are equivalent*

- *$Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$,*
- *$Ax = 0$ has a unique solution,*
- *the row/column vectors are all linearly independent,*
- *the matrix has full rank, i.e.* $\mathrm{Rank}(A) = n$,
- *the nullity of the matrix is zero,*
- *the determinant is non-zero,* $\det(A) \neq 0$.

A special type of matrix is a **upper/lower triangular matrix**. For upper triangular matrices, every entry below the diagonal is zero. For lower triangular matrices, every entry above the diagonal is zero. For example, the following is upper triangular

$$A = \begin{bmatrix} a_{1,1} & \cdots & \cdots & a_{1,n} \\ & \ddots & & \vdots \\ & 0 & \ddots & \vdots \\ & & & a_{n,n} \end{bmatrix}$$

For such a matrix, we can find the determinant by going along the bottom row. All the elements are zero except the last, so plugging this into the determinant definition we have

$$a_{n,n} \cdot \det(A) = \det \begin{bmatrix} a_{1,1} & \cdots & \cdots & a_{1,n} \\ & \ddots & & \vdots \\ & 0 & \ddots & \vdots \\ & & & a_{n-1,n-1} \end{bmatrix}.$$

Repeating this inductively, we find that

$$\det(A) = a_{n,n} \cdot a_{n-1,n-1} \cdots a_{1,1}.$$

So the determinant of a triangular matrix is is the product of the entries along the diagonal.

Definition 3.18 (Transpose of a matrix)**.** Given a matrix $A \in \mathbb{R}^{m\times n}$, the **matrix transpose** of A is and $n \times m - matrix$ given by

$$A^T = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{bmatrix}.$$

i.e. the matrix with entries swapped over the diagonal.

We quickly look at some properties of the transpose

Lemma 3.19 (Properties of the Transpose)**.** *Let $A \in \mathbb{R}^{m\times n}$. The following hold*

- $(A^T)^T = A$,
- $(A+B)^T = A^T + B^T$ *for any* $B \in \mathbb{R}^{m\times n}$,
- $(\alpha A)^T = \alpha A^T$ *for all* $\alpha \in \mathbb{R}$,
- $(AB)^T = B^T A^T$ *for any* $B \in \mathbb{R}^{n\times p}$.

With that out of the way, we can continue to explore the determinant,

Lemma 3.20 (Properties of the Determinant)**.** *Let $A, B \in \mathbb{R}^{n\times n}$ be a square matrix, and $\alpha \in \mathbb{R}$. The determinant satisfies the following properties*

- $\det(A^T) = \det(A)$,
- $\det(AB) = \det(A)\det(B)$,
- $\det(\alpha A) = \alpha^n \det(A)$.

Note. In general the determinant of the the sum of two matrices is not the sum of the determinants, i.e.

$$\det(A+B) \neq \det(A) + \det(B).$$

For example, consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$\det(A) = \det(B) = 1,$$

but

$$\det(A+B) = 0.$$

Another logical question to ask is: **How do elementary row operation affect the determinant?**

If we know how these work, then one way of calculating the determinant would be to reduce the matrix to a triangular matrix and calculate the determinant of that. Well, the three operations do the following:

- Interchanging rows multiplies the determinant by -1,
- adding a multiple of one row to another does not change the determinant,
- Scaling a row by a constant $\alpha \neq 0$ multiplies the determinant by α.

3.4 Eigenvalues

The first goal of this section is to answer a simple question. Given some matrix $A \in \mathbb{R}^{n\times n}$ with some associated linear map $T : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax$, for what vectors is applying this mapping equivalent to multiplying by a scalar? Geometrically, these are vectors for which the mapping preserves the direction of the vector, but not the magnitude in general.

Such vectors are special.

Definition 3.21 (Eigenvectors and Eigenvalues)**.** Let $A \in \mathbb{R}^{n\times n}$ be some square matrix. A vector $x \in \mathbb{R}^n$ with $x \neq 0$ is an **eigenvector** of A if

$$Ax = \lambda x,$$

for some $\lambda \in \mathbb{C}$. This λ is then called an **eigenvalue**.

Note. λ can be a complex value in this definition. We will see in an example later on that by restricting ourselves to $\lambda \in \mathbb{R}$ we will miss out on eigenvalues. If $\lambda \in \mathbb{C}$ is an eigenvalue, then its conjugate $\bar{\lambda}$ will also be an eigenvalue.

There are finite eigenvalues for a matrix A, but for each eigenvalue there may be many associated eigenvectors.

How do you find the eigenvalues? The following calculation reveals something important.

$$Ax = \lambda x \iff Ax - \lambda x = 0 \iff (A - \lambda I)x = 0.$$

As we require that $x \neq 0$, x must be a non-zero element of the nullspace $\mathrm{Null}(A - \lambda I)$. This means the nullity of $A - \lambda I$ is non-zero, and thus by Lemma 3.20

$$\det(A - \lambda I) = 0.$$

Hence to find the eigenvalues of A, we have to find the roots of the polynomial

$$p(\lambda) = \det(A - \lambda I).$$

This is called the **characteristic polynomial** of A.

By the fundamental theorem of algebra, we can write

$$p(\lambda) = (\lambda - \lambda_1)^{\mu_1}(\lambda - \lambda_2)^{\mu_2} \cdots (\lambda - \lambda_k)^{\mu_k},$$

where $k \leq n$ and

$$\mu_1 + \mu_2 + \cdots + \mu_k = n.$$

These λ_i, $i = 1, \ldots, k$ are the eigenvalues of A, and the μ_i are their **(algebraic) multiplicities**.

The **geometric multiplicity** γ_i of some eigenvalue λ_i, $i = 1, \ldots, k$ of A is the number of linearly independent eigenvectors associated with that eigenvalue. i.e. it is

$$\gamma_i = \dim(\mathrm{Null}(A - \lambda_i I)).$$

We call the vector space $E_i \coloneqq \mathrm{Null}(A - \lambda_i I)$ the **eigenspace** of the eigenvalue λ_i.

In general $\gamma_i \leq \mu_i$. If $\gamma_i < \mu_i$ then we have something called **generalised eigenvectors**, which we will not consider here.

Let's now test the theory in an example.

Example 3.22. Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We first solve

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 & 0 \\ 4 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = -(\lambda - 1)(\lambda - 4)(\lambda + 4).$$

Finding the roots of p reveals three eigenvalues with multiplicity 1,

$$\lambda_1 = 1, \ \lambda_2 = 4, \ \lambda_3 = -4.$$

To find their eigenvalues, we solve the problem

$$(A - \lambda_i I)x_i = 0,$$

for each $i = 1, 2, 3$. For the first, we have

$$\begin{bmatrix} 1 & 3 & 0 \\ 4 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = 0,$$

and thus we find after solving this problem that for any $t \in \mathbb{R}$

$$\begin{bmatrix} \frac{1}{5} \\ -\frac{1}{15} \\ 1 \end{bmatrix} t$$

is an eigenvector. Thus the eigenspace for $\lambda_1 = 1$ is

$$E_1 = \operatorname{span}\left\{ \begin{bmatrix} \frac{1}{5} \\ -\frac{1}{15} \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 15 \end{bmatrix} \right\}.$$

Solving the problem for the other two eigenvalues, we find their respective eigenspaces are

$$E_2 = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \right\} \text{ and } E_3 = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

Example 3.23. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The characteristic polynomial is given by

$$p(\lambda) = (\lambda - 1)^2.$$

So we have one eigenvalue $\lambda = 1$ with algebraic multiplicity 2. Solving for the eigenvectors

$$0 = (A - I)x = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x,$$

we find the eigenspace is

$$E = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

The geometric multiplicity of this eigenvalue, i.e. the dimension of its eigenspace, is 1. So its geometric multiplicity is smaller than the algebraic multiplicity, $1 < 2$. In this case we could talk about generalised eigenvectors.

3.5 Inner Products, Orthogonal Bases and Complements

When talking about vectors in $\mathbb{R}^n$ we mentioned a operation relating to the angle between two vectors, the dot/inner product:

$$x \cdot y = \langle x, y \rangle = \sum_{i=1}^{n} x_i, y_i, \qquad (x, y \in \mathbb{R}^n).$$

In this chapter we are going to be sticking to the space $\mathbb{R}^n$. However, many of the definitions can also be made for an arbitrary vector space V, assuming an inner product is defined on it.

Definition 3.24 (Inner Product)**.** Let V be an vector space (over $\mathbb{R}$). V is an **inner product space** if there exists a mapping $\langle \cdot, \cdot \rangle : V \to \mathbb{R}$, called its **inner product**, satisfying three conditions

1. **Linearity in the first argument**: For any $x, y, z \in V$ and $\alpha \in \mathbb{R}$:

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle,$$

$$\langle \alpha x, z \rangle = \alpha \langle x, z \rangle.$$

2. **Symmetry**: $\langle x, y \rangle = \langle y, x \rangle$ for any $x, y \in \mathbb{R}$.

3. **Positive Definiteness**: $\langle x, x \rangle > 0$ for any non-zero $x \in \mathbb{R}$.

Two vectors are **orthogonal** if their dot product was zero: $x \cdot y = 0$.

Another property we had was the length of a vector,

$$|x| = \sqrt{x \cdot x}, \qquad (x \in \mathbb{R}^n).$$

We can define this similarly for an arbitrary inner product space if we wish.

Note. One way of writing the dot product of two column vectors $x, y \in \mathbb{R}^n$ is as

$$x \cdot y = x^T y = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i.$$

This leads to the property: For any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$,

$$(Ax) \cdot y = (Ax)^T y = x^T A^T y = x \cdot (A^T y).$$

where we have used that $(Ax)^T = x^T A^T$ from Lemma 3.19.

An important type of vector is one with length 1. We call these **normalised vectors**. We **normalise** a non-zero vector $x \in \mathbb{R}^n$ by converting it to a new vector $\hat{x}$ with length 1, given by

$$\hat{x} = \frac{x}{|x|}.$$

This vector has the same direction as x, but it has length 1.

With these properties in mind, we have our first definition.

Definition 3.25 (Orthogonal/Orthonormal Bases)**.** s Let $V \subseteq \mathbb{R}^n$ be some subspace of $\mathbb{R}^n$, with a basis $\{v_1, \ldots, v_k\}$ where $k \leq n$. We call this an **orthogonal basis** if

$$v_i \cdot v_j = 0 \quad (i, j \in \{1, \cdots, n\}, i \neq j).$$

If in addition we have the property

$$|v_i| = 1, \quad (i = 1, \ldots, n),$$

then we call it an **orthonormal basis**. In such a case

$$v_i \cdot v_j = \delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Example 3.26.

- The standard basis for $\mathbb{R}^n$, $\{e_1, \ldots, e_n\}$ is orthonormal.
- The basis $\{(1, 2), (-2, 1)\}$ is orthogonal.
- We find

  $$|(1, 2)| = \sqrt{5} = |(-2, 1)|$$

  and so $\left\{\frac{1}{\sqrt{5}}(1, 2), \frac{1}{\sqrt{5}}(-2, 1)\right\}$ is an orthonormal basis.

3.5.1 Gram-Schmidt-Orthogonalisation

A logical question to ask now is the following: **If we have a basis for a vector space V, can we generate an orthogonal/orthonormal basis?**. Before we answer this, we introduce the projection of a vector.

Definition 3.27 (Projection)**.** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be two vectors. The **projection** of $\mathbf{v}$ onto $\mathbf{u}$ is given by

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}.$$

Geometrically, the vector $\mathbf{v}$ can be written as combination of two vectors. One vector parallel to $\mathbf{u}$ and one perpendicular to $\mathbf{u}$. The projection outputs the parallel part. See Figure 1.

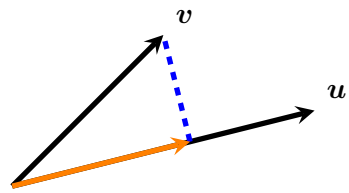

Figure 1: The orange vector is the projection of a vector $\mathbf{v}$ onto a a vector $\mathbf{u}$.

Note. Notice that for any vector $\mathbf{w}$

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) \cdot \mathbf{w} = C(\mathbf{u} \cdot \mathbf{w}).$$

where $C = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}$ is a constant. Also

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v}.$$

The process of creating an orthogonal basis is called **Gram-Schmidt-Orthogonalisation**. The algorithm is as follows:

Suppose we have a basis $\{v_1, \ldots, v_n\}$ for some vector space V in $\mathbb{R}^n$. We define the vectors $\{\hat{v}_1, \ldots \hat{v}_n\}$

$$\begin{aligned}
\hat{v}_1 &= v_1, \\
\hat{v}_2 &= v_2 - \text{proj}_{\hat{v}_1}(v_2), \\
\hat{v}_2 &= v_3 - \text{proj}_{\hat{v}_1}(v_3) - \text{proj}_{\hat{v}_2}(v_3), \\
&\vdots \\
\hat{v}_n &= v_n - \sum_{k=1}^{n-1} \text{proj}_{\hat{v}_k}(v_n).
\end{aligned}$$

Then the set $\{\hat{v}_1, \ldots \hat{v}_n\}$ is an orthogonal basis for V. Normalising each of the vectors creates an orthonormal basis for V.

3.5.2 Orthogonal Complements

The orthogonal complement of a subspace contains all the elements of the vector space orthogonal to all elements of the subspace.

Definition 3.28 (Orthogonal Complement)**.** Let V be a subspace of $\mathbb{R}^n$. The **orthogonal complement** of V is the set

$$V^{\perp} = \{x \in \mathbb{R}^n \mid \langle x, y \rangle = 0 \text{ for all } y \in V\}.$$

This set has some nice results

Lemma 3.29. *Let V be a subspace of $\mathbb{R}^n$, and $A \in \mathbb{R}^{m\times n}$. Then*

- $\dim(V) + \dim(V^\perp) = n$,
- $\mathrm{Row}(A)^\perp = \mathrm{Null}(A)$.

Note. Combining these two results, we get

$$n = \dim(\mathrm{Row}(A)) + \dim(\mathrm{Row}(A)^\perp) = \dim(Rowt(A)) + \dim(Nullt(A)),$$

i.e. the rank-nullity theorem.

3.6 Some Special Matrices

We are now going to introduce two important classes of matrices, which will play a role in an important process we will introduce, diagonalisation.

Definition 3.30 (Symmetric Matrix)**.** We say a square matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if the enteries are symmetric over the diagonal, or rather

$$A^T = A.$$

For example, the matrix

$$\begin{bmatrix} 1 & 6 & 0 \\ 6 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

is symmetric.

These matrices have some very nice properties

Lemma 3.31 (Properties of Symmetric Matrices)**.** *Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric, and $\alpha \in \mathbb{R}$. The following hold,*

- *A only has real eigenvalues,*
- *$A + B$ and αA are symmetric,*
- *If the inverse A^{-1} exists, it is symmetric,*
- *The eigenvectors are orthogonal.*

Proof of the final property. Let x_1, x_2 be two eigenvectors, with two distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$. Then

$$\lambda_1\langle x, y\rangle = \langle \lambda_1 x, y\rangle = \langle Ax, y\rangle = \langle x, A^T y\rangle = \langle x, Ay\rangle = \langle x, \lambda_2 y\rangle = \lambda_2\langle x, y\rangle,$$

and so

$$(\lambda_1 - \lambda_2)\langle x, y\rangle = 0,$$

and as we have assumed $\lambda_1 \neq \lambda_2$, we must have $\langle x, y\rangle = 0$, as required. □

Suppose $A \in \mathbb{R}^{n\times n}$ has n eigenvectors (i.e. all the algebraic multiplicities equal the geometric multiplicities, so no generalised eigenvectors). We can then construct an orthonormal basis of $\mathbb{R}^n$ consisting of the eigenvectors of A, via the use of Gram-Schmidt-Orthogonalisation.

Via the use of something called the Spectral Theorem, we can say this is always possible for symmetric matrices.

Theorem 3.32. *Let $A \in \mathbb{R}^{n\times n}$ by symmetric. Then there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.*

We introduce one final special matrix.

Definition 3.33 (Orthogonal Matrix)**.** Let $A \in \mathbb{R}^{n\times n}$. A is **orthogonal** if its transpose is its inverse, that is

$$A^{-1} = A^T.$$

If $A \in \mathbb{R}^{n\times n}$ is orthogonal,

$$AA^T = A^T A = I.$$

Then the vectors for the rows and columns are orthonormal vectors. To see this, we have

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1\cdot\mathbf{c}_1 & \cdots & \mathbf{r}_1\cdot\mathbf{c}_n \\ \vdots & \ddots & \vdots \\ \mathbf{r}_n\cdot\mathbf{c}_1 & \cdots & \mathbf{r}_n\cdot\mathbf{c}_n \end{bmatrix} = \begin{bmatrix} 1 & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & 1 \end{bmatrix}.$$

The columns of A then form an orthonormal basis of $\mathbb{R}^n$.

One final thing to note is that

$$|Ax|^2 = \langle Ax, Ax\rangle = \langle x, A^T Ax\rangle = \langle x, x\rangle = |x|^2$$

i.e. orthogonal transformations preserve vector length.

3.7 Similarity and Diagonalisation

Large matrices can be difficult or expensive to make calculations with. The goal now is find a simpler form of a complicated matrix which is easier to work with.

Any vector in a finite dimensional vector space V with a basis $\{v_1, \dots v_n\}$ can be written as a unique linear combination of the basis elements. The coefficients of this linear combination can be represented by a vector $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$. A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ can then be defined with respect to this basis.

This thought process is important. So far, we have often thought of the vectors in $\mathbb{R}^n$ in linear mappings as exactly what they are, just vectors in $\mathbb{R}^n$. In actual fact, these vectors represent a linear combination of basis elements. The basis we have so far been using is the typical basis of $\mathbb{R}^n$, $\{e_1, \dots, e_n\}$.

With this in mind, we introduce a definition.

Definition 3.34 (Matrix Similarity)**.** A matrix $A \in \mathbb{R}^{n\times n}$ so **similar** to another matrix $B \in \mathbb{R}^{n\times n}$ if there exists a non-singular (invertible) matrix $P \in \mathbb{R}^{n\times n}$ such that

$$A = P^{-1}BP.$$

Note. Two similar matrices represent the same linear map under different bases.

Suppose we have one linear mapping $T(x) = Bx$ which is a mapping on standard basis vectors, i.e. the x coordinate (x_1, x_2, x_3) represents the linear combination

$$x_1e_1 + x_2e_2 + x_3e_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Let $A \in \mathbb{R}^{n\times n}$ be a similar matrix. $\hat{T}(\hat{x}) = A\hat{x}$ is the same linear mapping where the $\hat{x}$ coordinate represents the linear combination with basis vectors $b_i = P^{-1}e_i$,

$$\hat{x}_1b_1 + \hat{x}_2b_2 + \hat{x}_3b_3.$$

Lemma 3.35. *Similar matrices have the same characteristic polynomial.*

Proof. This follows from the following calculation

$$\begin{aligned}
\det(A - \lambda I) &= \det(P^{-1}BP - \lambda PP^{-1}) \\
&= \det(P^{-1}(BP - \lambda P)) \\
&= \det(P^{-1}(B - \lambda I)P) \\
&= \det(P^{-1})\det(B - \lambda I)\det(P) \\
&= \det(P^{-1})\det(P)\det(B - \lambda I) \\
&= \det(P^{-1}P)\det(B - \lambda I) \\
&= \det(I)\det(B - \lambda I) \\
&= \det(B - \lambda I).
\end{aligned}$$

□

Lemma 3.36. *If x is an eigenvector of B, $y = P^{-1}x$ is an eigenvector of A.*

Our goal now is as follows: Given some linear mapping defined by a vector B, we wish to find a similar matrix given by a diagonal matrix A. It will be easier to calculate Ax in the new basis than Bx in the old one. Solving $Ax = b$ will thus be easier.

Suppose $B \in \mathbb{R}^{n\times n}$ has a basis of n eigenvectors (algebraic multiplicity = geometric multiplicity) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Define

$$P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}. \tag{3}$$

Then

$$\begin{aligned}
BP = B\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} &= \begin{bmatrix} B\mathbf{v}_1 & \cdots & B\mathbf{v}_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}^T \\
&= \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = PD
\end{aligned}$$

where

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \tag{4}$$

So we can summarise

Lemma 3.37. *Let $B \in \mathbb{R}^{n\times n}$ be a matrix with n eigenvectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ with associated eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. Then the matrix P defined by (3), and the matrix D defined by (4) satisfies*

$$B = PDP^{-1},$$

i.e. B and D are similar.

We know from the previous section that we can always find an orthonormal basis of n eigenvectors, so we have the following result.

Corollary 3.38. *Symmetric matrices are diagonalisable, and one can choose an orthogonal matrix P, i.e. $P^{-1} = P^T$.*

We have a diagonal matrix, what can we do with it? Given $B \in \mathbb{R}^{n\times n}$ similar to a diagonal matrix D, then for any $N \in \mathbb{N}$,

$$B^n = (PDP^{-1})^N = PDP^{-1}PDP^{-1} \dots P^{-1}PDP^{-1} = PDD \dots DP^{-1} = PD^NP^{-1}.$$

and

$$D^N = \begin{bmatrix} \lambda_1^N & & 0 \\ & \ddots & \\ 0 & & \lambda_n^N \end{bmatrix}. \tag{5}$$

Note. If B does not have enough eigenvalues, then we can write $B = P^{-1}JP$, for "almost" diagonal matrix J. This is called Jordan canonical form.

3.8 The Matrix Exponential

We end now with a slight diversion looking at a useful matrix for ordinary differential equations. The definition for the normal exponential function for $t \in \mathbb{R}$ is

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

For a matrix $A \in \mathbb{R}^{n\times n}$ we define

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n, \qquad \text{where } A^0 = I.$$

Suppose B is diagonalisable where $B = PDP^{-1}$, with D diagonal, then

$$e^B = e^{PDP^{-1}} = \sum_{n=0}^{\infty} \frac{1}{n!}(PDP^{-1})^n = \sum_{n=0}^{\infty} \frac{1}{n!} PD^n P^{-1} = P\left(\sum_{n=0}^{\infty} \frac{1}{n!} D^n\right) P^{-1} = Pe^D P^{-1}.$$

Further

$$e^D = \begin{bmatrix} e^{d_1} & & 0 \\ & \ddots & \\ 0 & & e^{d_n} \end{bmatrix}.$$

4 Ordinary Differential Equations

4.1 Continuity and Differentiation

Before beginning, we introduce some of the fundamental concepts we need to know.

Let $I \subset \mathbb{R}$ be an open interval, and $f : I \to \mathbb{R}$ be defined on the interval I.

Definition 4.1 (Continuity)**.** Let $s \in I$. f is **continuous at s** if its limit $\lim_{t\to s} f(t)$ exists and

$$f(s) = \lim_{t\to s} f(t).$$

We say f is **continuous** if it is continuous at all $s \in I$.

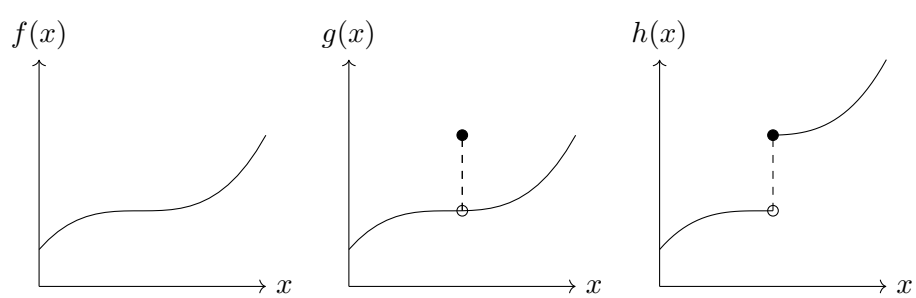

Figure 2: From right to left: f is a continuous function, g is continuous except at a single point, and h is a discontinuous function.

Definition 4.2 (Differentiability)**.** f is **differentiable at a point** $s \in I$ if the limit $\lim_{t\to s} \frac{f(t)-f(s)}{t-s}$ exists, and we call

$$\frac{df}{ds}(s) = f'(s) = \lim_{t\to s}\left[\frac{f(t)-f(s)}{t-s}\right] \qquad \left(= \lim_{h\to 0} \frac{f(t+h)-f(h)}{h}\right).$$

f is **differentiable** if it is differentiable at every point $s \in I$.

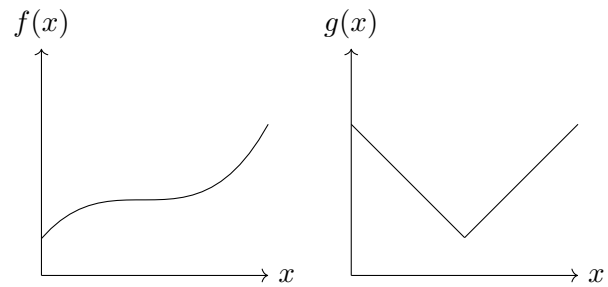

Figure 3: From right to left: f is a differentiable function, g is continuous function, yet it is not differentiable.

Lemma 4.3. *If f is differentiable, it must be continuous. The opposite is not true in general.*

We now introduce one final notion of continuity.

Definition 4.4 (Lipschitz Continuity)**.** f is Lipschitz continuous if there exists an $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|,$$

for all $x, y \in I$.

Lipschitz continuity is a restriction on the growth of function. It states the slope of a function can never grow too large.

Example 4.5. We will considers functions mapping $\mathbb{R}$ to $\mathbb{R}$.

- $f(t) = \sin(t)$ is continuous, differentiable, and Lipschitz continuous. As, by the mean value theorem,

  $$|\sin(t) - \sin(s)| \leq |\cos(c)||t - s| \leq |t - s|$$

  where c lies between 0 and π.

- $g(t) = |t|$ is continuous and Lipschitz continuous, but not differentiable. From the reverse triangle inequality

  $$||t| - |s|| \leq |t - s|.$$

- $h(t) = |t|^{\frac{1}{2}}$ is continuous, differentiable for all $t \neq 0$, and not Lipschitz continuos.

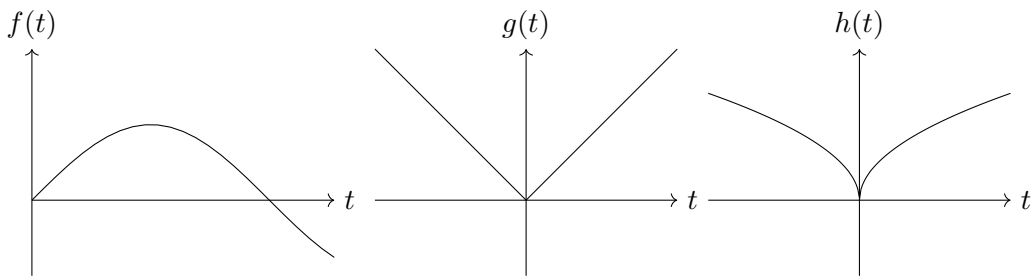

Figure 4: Plots of f, g and h from Example 4.5

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4.2 Differential Equations and Dynamical Systems

Let's start working on the object that is the focus of the second half of this course, differential equations.

Definition 4.6 (Ordinary Differential Equation)**.** An **Ordinary Differential Equation** (ODE) is an equation involving the derivative of an unknown function,

$$F(x, u(x), u'(x), \ldots, u^{(n)}(x)) = 0, \tag{6}$$

where u here is the unknown function.

We say the ODE is **order n** if the highest derivative is of order n.

The ODE is **linear** if F is a linear combination of derivatives,

$$F(x, u(x), u'(x), \ldots, u^{(n)}(x)) = r(x) + \sum_{i=0}^{n} \alpha_i u^{(i)}(x)$$

where here $r : \mathbb{R} \to \mathbb{R}$ is some given function and α_i are constants.

Example 4.7. We will look at some examples of first and second order ODEs.

- $y'(x) - \alpha y(x) = 0$ where α is some constant in $\mathbb{R}$. This is a linear first order differential equation.
- $u''(t) + \omega^2 u(t) = \cos \omega t$ where ω is some constant in $\mathbb{R}$ is a second order ODE.

The next crucial step is define what form we require the solutions to take.

Definition 4.8 (ODE solution)**.** Let I be some interval in $\mathbb{R}$. A function $u : I \to \mathbb{R}$ is a **solution** to the ODE (6) if it satisfies the equation for all x in I, and it is a continuous, n-times differentiable function

We can couple a group of ODEs together to form a system.

Definition 4.9 (ODE system)**.** A **system of ODEs** is a set of equations written in the form

$$\dot{u}(t) = f(u(t)),$$

where $\dot{u}$ refers to the time derivative, $f : \mathbb{R}^n \to \mathbb{R}^n$, and $u(t) \in \mathbb{R}^n$.

Example 4.10. An example of such an ODE system would be

$$\dot{x}(t) = -0.2x(t) + 0.3y(t)$$
$$\dot{y}(t) = 0.2x(t) - 0.3y(t)$$

which we can rewrite as in matrix form as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The next definition covers a particular interpretation of an ODE

Definition 4.11 (Dynamical System)**.** A **Dynamical System** is an ODE of the form

$$\dot{x}(t) = f(x(t)).$$

These are systems which describe the position of a particle x in space at time t. The function $f : \mathbb{R}^n \to \mathbb{R}^n$ is the velocity of the particle at position x.

Example 4.12 (Converting from a 1 dimensional ODE to a system)**.** We are given an ODE of the form

$$x^{(n)}(t) = F(t, x(t), \ldots, x^{(n-1)}(t)).$$

Our goal is to convert this to a system of ODEs.

We define some new functions $x_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$ given by

$$x_1(t) = x(t),$$
$$x_2(t) = \dot{x}(t) = \dot{x}_1(t),$$
$$x_i(t) = x^{(i-1)}(t) = \dot{x}_{i-1}(t), \qquad i = 3, \ldots, n.$$

This leads to a system of the form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ F(t, x_1(t), \ldots, x_n(t)) \end{bmatrix}.$$

Suppose now that we have an ODE of the form

$$x^{(n)}(t) = \sum_{i=0}^{n-1} \alpha_i x^{(i)}(t) + r(t),$$

where $r : \mathbb{R} \to \mathbb{R}$ is some given function and the α_i, $i = 1, \ldots, n-1$, are constants. Then our system looks like

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ \alpha_1 & \cdots & \alpha_{n-2} & \alpha_{n-1} \end{bmatrix}}_{:=A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}}_{:=\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ r(t) \end{bmatrix}}_{:=\mathbf{b}}.$$

or in other words a system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}.$$

Example 4.13. Consider the second order ODE

$$y''(t) - 2y'(t) + 4y^3(t) = 0.$$

Set

$$x_1(t) = y(t), \text{ and } x_2(t) = y'(t),$$

then we can write this ODE in the form of a system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ 2x_2(t) - 4x_1^3(t) \end{bmatrix}.$$

Example 4.14. In this example we will remove the time dependence of the right hand side function of an ODE. Consider

$$y'(t) = f(t, y(t)),$$

where we have

$$f(t, y(t)) = 3 + ty(t).$$

Define

$$x_1(t) = y(t), \text{ and } x_2(t) = t,$$

then we can write this ODE in the form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 3 + x_1(t)x_2(t) \\ 1 \end{bmatrix}.$$

This ODE is now a system with the form

$$\dot{\mathbf{x}}(t) = F(\mathbf{x}(t)).$$

4.3 Existence and Uniqueness

Lets now consider an ODE and its associated solution. We quickly notice an issue.

Example 4.15. Consider the ODE

$$y'(t) = 3y(t).$$

A solution for this ODE is

$$y(t) = e^{3t}.$$

Indeed, the derivative is $y'(t) = 3e^{3t} = 3y(t)$. However, one can also show that

$$y(t) = \alpha e^{3t}$$

is a solution to this problem, for any choice of $\alpha \in \mathbb{R}$.

The problem we have is that there are too many possible solutions for this ODE. That is to say, solutions are not necessarily unique! To resolve this issue, we must introduce another restriction to form our problem, leading to the following definition.

Definition 4.16 (Initial Value Problem)**.** An **initial value problem** (IVP) is an ODE supplied with some initial data $x_0 \in \mathbb{R}$, given by

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0. \end{cases} \tag{IVP}$$

Let a function $x : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is some interval containing 0, be a function. x is a solution to (IVP) if is continuous, differentiable, satisfies $x(0) = x_0$, and

$$\dot{x}(t) = f(x(t)).$$

A reasonable question to ask is: How do we know for sure that the solution exists and is unique? We have two theorems for this, the first pertaining to just existence, and the second to both uniqueness and existence.

Theorem 4.17 (Peano's Existence Theorem)**.** *Let $D \subset \mathbb{R}^2$ be an open set and $f : D \to \mathbb{R}$ be a continuous function. Then the initial value problem*

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases}$$

with $(t_0, x_0) \in D$ has a solution on some open interval I containing x_0.

The solution need not be unique.

Theorem 4.18 (Picard-Lindelöf Existence and Uniqueness Theorem)**.** *Suppose f be locally Lipschitz (Lipschitz on some open set containing 0), then there exists a unique solution of* (IVP) *on some open set containing* 0 *(i.e. on a set* $[0-\epsilon, 0+\epsilon]$ *for some* $\epsilon > 0$*).*

These theorems are important because they tell us of the existence and/or uniqueness of solutions, but they do not tell use what the solution is.

Example 4.19. Lets make use of these theorems in the following three examples. We look at the problem $\dot{\mathbf{x}} = f(x)$ for the following f,

- $f(x) = |x|^{\frac{1}{2}}, x(0) = 0$. We have already mentioned, this function is not Lipschitz. Peano's theorem does give existence though. Both $x_1(t) = 0$ and $x_2(t) = \frac{1}{4}sgn(t)t^2$ are solutions defined for all $t \in \mathbb{R}$.

- $f(x) = x^2, x_0 = 1$. This is Lipschitz locally, but not globally Lipschitz. Peano's theorem and Picard-Lindelöf apply. This has the unique solution

$$x(t) = (1-t)^{-1}$$

  defined on the maximal interval $I = (-\infty, 1]$.

- $f(x) = x, x_0 = 1$. This is Lipschitz (globally), so both theorems apply. This has the unique solutions

$$x(t) = e^t, \qquad (t \in \mathbb{R}).$$

4.4 Phase Planes and Phase Portraits

Can we determine the behaviour of the ODE

$$\dot{x}(t) = f(x(t)),$$

without solving the problem explicitly? How does the solution depend on the initial data?

To move towards and answer to this question, we introduce the following definition.

Definition 4.20 (Equilibrium Point)**.** If $f(t_0) = 0$, t_0 is an **equilibrium point**. These are points at which $\dot{x}(t_0) = 0$, so $x(t)$ stays at x_0 for all t.

Example 4.21. Consider $f(x) = -x(x-a)$, for some $a \in \mathbb{R}$. Then $f(x) = 0$ for $x = 0$ and $x = a$. Thus both of these are equilibrium points. Assume $a > 0$.

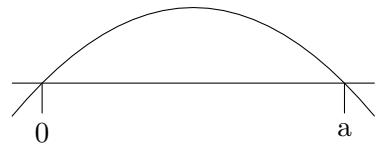

Figure 5: Plot of f from Example 4.21.

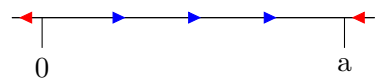

Figure 6: Plot of increasing and decreasing regions for the solution given some initial data, from Example 4.21.

What about if we are outside of these equilibrium points? We wish to examine the behaviour in these situations.

Upon plotting f, we see that there are 3 regions, in which there 2 different behaviours are exhibited. For a choice of $x_0 \in \mathbb{R}$ we have for t near 0 that

- $x_0 < 0$: Then $\dot{x}(t) \approx f(x_0) < 0$, so we are moving to the left,
- $0 < x_0 < a$: Then $\dot{x}(t) \approx f(x_0) > 0$, so we are moving to the right,
- $a < x_0$: Then $\dot{x}(t) \approx f(x_0) < 0$, so we are moving to the left,

These regions are plotted in Figure 6.

We name the equilibrium points based on the behaviour of the solutions near these points:

- the point $x_0 = 0$ is an **unstable** equilibrium point. If the solution starts near here, it will move away over time,
- the point $x_0 = a$ is an **stable** equilibrium point. If the solution starts near here, it will towards this point over time.

We explore another example, this time of a second order ODE. In this situation we plot a phase portrait.

Example 4.22. In our core example for a phase portrait we will be modelling the movement of an ideal pendulum under gravity. We assume that the mass of the pendulum is one. We will calculate the angle θ at time t. This is modelled using a second order differential equation,

$$\ddot{\theta} + \sin\theta = 0.$$

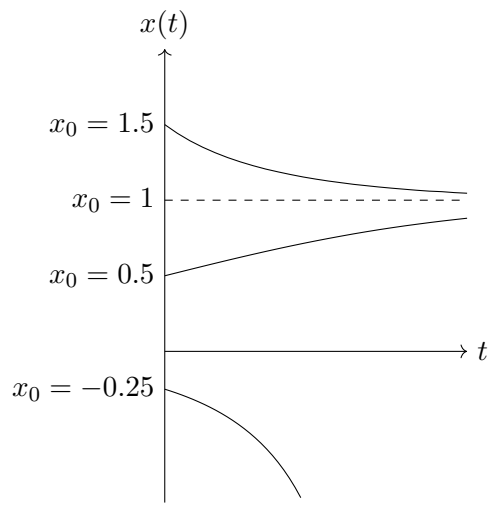

Figure 7: Plot of $x(t)$ from Example 4.21 for different initial data.

We convert this into an ODE system, setting $x_1 = \theta$ and $x_2 = \dot{\theta}$, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin(x_1) \end{bmatrix} := f(x_1, x_2).$$

The first question we ask: **What are the equilibrium points?**

Physically, these are the points when either the pendulum is straight up, or it is straight down. Should it be straight down, if you pushed it slightly it would return to the same position. If it was straight up, it would eventually land on being straight down. We name them thus:

- If $x_1 = 2n\pi$ and $x_2 = 0$, then this is a **stable** equilibrium point,
- If $x_1 = 2n(\pi + 1)$ and $x_2 = 0$, then this is a **unstable** equilibrium point,

here $n \in \mathbb{N}$ is any natural number.

The totally energy in this system is given by the sum of the kinetic and potential energy. This is given by

$$E(x_1, x_2) = \frac{1}{2}x_2^2 - \cos x_1.$$

Taking the derivative with respect to time, we have

$$\frac{d}{dt}(E(x_1, x_2)) = \dot{x_2}x_2 + \dot{x_1}\sin x_1 = -x_2\sin(x_1) + x_2\sin(x_1) = 0,$$

i.e. we have the preservation of energy in time.

We thus see solutions to this problem lie on level-sets (level-curves). These are curves which correspond to some initial energy in the system, which is then preserved for all time.

The energy at $(\pi, 0)$ is given by

$$E(\pi, 0) = \frac{1}{2}0^2 - \cos(\pi) = 1,$$

and this separates the two types of solutions.

We now draw the phase portraits. To do so, we can plot the vector $f(x_1, x_2)$ for each point on our two axis, creating a phase plane. This is seen in Figure 8.

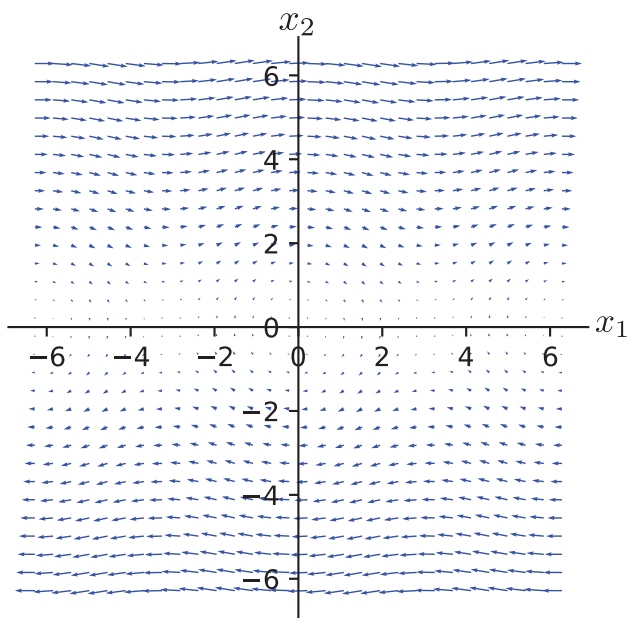

Figure 8: Drawing the phase plane.

After this, we draw the curves that follow these arrows, as seen in Figure 9

4.5 Numerics - Eulers Method

In this section we will be exploring a numerical method for the initial value problem

$$\begin{cases} \dot{x}(t) = f(x(t)), x(0) = x_0. \end{cases}$$

We assume f has sufficient assumptions (i.e. those of Picard-Linelöf) such that there is a unique solution to this problem.

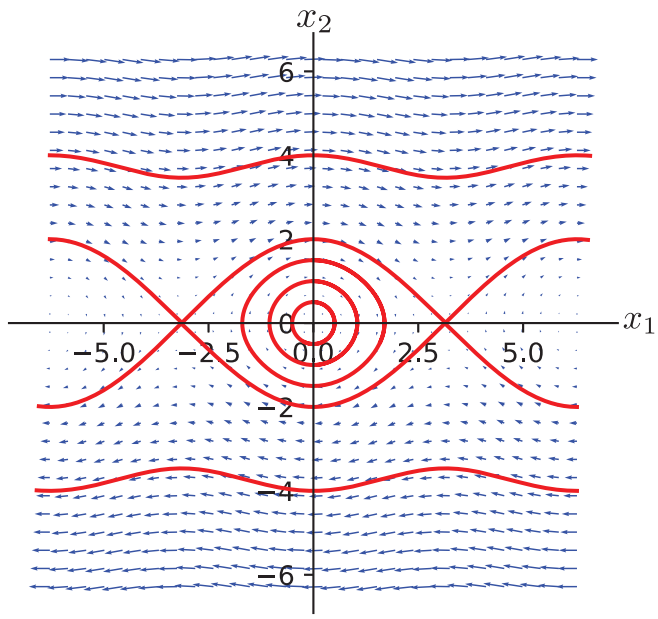

Figure 9: Some of the phase curves for the pendulum.

Euler's method is the simplest numerical method for ODEs. To begin, we define some step size $h > 0$. We then discretize time, setting

$$t_0 = 0, \qquad t_n = t_{n-1} + h,$$

where $n \in \mathbb{N}$.

Then, directly from the definition of the derivative,

$$\dot{x}(t_n) = \lim_{h \to 0} \left[\frac{x(t_n + h) - x(t_n)}{h} \right] \approx \frac{x(t_{n+1}) - x(t_n)}{h}.$$

We define the approximate value of the function at time step t_n as

$$x_n \approx x(t_n).$$

From this, using the fact that $\dot{x} = f(x)$, we define the method by

$$\frac{x_{n+1} - x_n}{h} = f(x_n),$$

or after rearranging

$$x_{n+1} = x_n + f(x_n).$$

This scheme of getting and approximation to the value of the function at the next time step given the value of the previous time step is Euler's method.

As we reduce the length of the time step, the approximation will improve. i.e. this approximation converges to the solution as $h \to 0$.

4.6 Analytical Method for 1st order ODEs

4.6.1 Separable Equations

A **separable ordinary differential equation** is an ODE of the form

$$f(y(t))\dot{y}(t) = g(t) \quad (\text{or } \dot{y}(t) = f(y(t))g(t)).$$

We begin by integrating this equation from 0 to t over the time variable,

$$\int_0^t f(y(s))\dot{y}(s)\ ds = \int_0^t g(s)\ ds.$$

Using the substitution $u = y(s)$, we have

$$\int_{y(0)}^{y(t)} f(u)\ du = \int_0^t g(s)\ ds.$$

Supposing now that F and G are the antiderivatives of f and g respectively, we get

$$F(y(t)) - F(y(0)) = G(t) - G(0).$$

At this point we can try to solve this equation for $y(t)$ to obtain an analytic expression for the solution.

The best way to approach separable equations is to follow a method demonstrated in the following examples.

Example 4.23. We consider the IVP

$$\dot{y} = ay, \quad y(0) = y_0 > 0.$$

where $a \neq 0$.

Rewritten this, we get

$$\frac{1}{y}\dot{y} = a.$$

The solution to this problem can then be found by integrating the left hand side with respect to y after dropping the $\dot{y}$, and integrating the right hand side with respect to t. So we have

$$\int_{y_0}^{y} \frac{1}{\tilde{y}} d\tilde{y} = \int_0^t a\ ds,$$

giving

$$\ln y - \ln y_0 = at,$$

and so

$$\ln y = \ln y_0 + at.$$

Taking the exponential of each side, we have

$$y = e^{\ln y} = e^{\ln y_0 + at} = e^{\ln y_0} e^{at} = y_0 e^{at}.$$

So we have the solution defined on all of $[0, +\infty)$,

$$y(t) = y_0 e^{at}.$$

Example 4.24. We now consider the problem

$$(1 + a^2 t^2) y' - y(1 - y) = 0,$$

where $a \neq 0$.

After rearranging, we have

$$\frac{1}{y(1-y)} y' = \frac{1}{1 + a^2 t^2}.$$

We calculate the partial fraction decomposition of the left, giving

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.$$

We find that

$$\int \left(\frac{1}{y} + \frac{1}{1-y} \right) \, dy = \ln y - \ln(1-y) + Const. \, .$$

Using the substitution $u = at$,

$$\int \frac{1}{1 + a^2 t^2} \, dt = \frac{1}{a} \int \frac{1}{1 + u^2} = \frac{1}{a} \arctan u + Const. = \frac{1}{a} \arctan at + Const. \, .$$

So we the solution is given by

$$\ln y(t) - \ln(1 - y(t)) = \frac{1}{a} \arctan at + A,$$

where A is some constant. After rearranging and taking the exponential,

$$\frac{y(t)}{1 - y(t)} = e^A e^{\frac{1}{a} \arctan at}.$$

After some work, we find that

$$y(t) = \frac{1}{1 + B e^{-\frac{1}{a} \arctan at}},$$

where B is some positive constant.

4.6.2 The Integrating Factor Method

We are now consider first order linear ODEs of the form

$$\dot{y}(t) + p(t)y(t) = g(t). \tag{7}$$

The **integrating factor** is

$$P(t) = \int_0^t p(s)\ ds.$$

It is best to learn the following method, rather to try and remember the formula given at the end.

We begin by multiplying (7) by $e^{P(t)}$, giving

$$e^{P(t)}\dot{y}(t) + e^{P(t)}p(t)y(t) = e^{P(t)}g(t),$$

which can be written as

$$\frac{d}{dt}\left(e^{P(t)}y(t)\right) = e^{P(t)}g(t).$$

Integrating this, we get

$$e^{P(t)}y(t) = \int e^{P(t)}g(t)\ dt + Const.,$$

and thus

$$y(t) = e^{-P(t)}\left(\int e^{P(t)}g(t)\ dt + Const.\right)$$

Example 4.25. Let's put this method to action by looking at the separable equation

$$y' = ay, \quad y(0) = y_0 > 0$$

where $a > 0$.

This can be written as

$$y' - ay = 0,$$

and we multiply this by e^{-at}, giving

$$e^{-at}y'(t) - e^{-at}ay(t) = 0,$$

and thus

$$\frac{d}{dt}\left(e^{-at}y(t)\right) = 0.$$

Integrating from 0 to t, we get

$$e^{-at}y(t) - y(0) = 0,$$

or rather

$$y(t) = y_0 e^{at},$$

as before.

Example 4.26. We now consider the problem

$$y'(t) - \cos(t)y(t) = e^{\sin t}, \qquad y(0) = 0.$$

We multiply by

$$e^{-\int \cos t \ dt} = e^{-\sin t},$$

giving

$$\frac{d}{dt}\left(e^{-\sin t} y(t)\right) = 1.$$

So, after integrating from 0 to 1, we get

$$e^{-\sin t} y(t) - y(0) = t,$$

and so, as $y(0) = 0$,

$$y(t) = te^{\sin t}.$$

4.7 Analytic Method for 2nd Order Linear ODEs

We now consider equations of the form

$$y''(t) + p(t)y'(t) + q(t)y(t) = r(t). \tag{8}$$

These are second order linear ordinary differential equations.

Example 4.27. Newtons 2nd law leads to a second order differential equation. Consider a block of mass m attached via a spring to a wall. Its center of mass is at position $x(t)$ at time t. We have Newtons 2nd

$$m\ddot{x} = \text{Forces}.$$

Here

- The force from the spring is given by $-kx(t)$ where k is some positive constant,
- the force due to friction is given by $-\mu\dot{x}(t)$ where μ is some constant,
- horizontal external forces are given by some function $-f(t)$.

So

$$m\ddot{x} = -kx - \mu\dot{x} - f,$$

or rather

$$\ddot{x} + \frac{\mu}{m}\dot{x} + \frac{k}{m}x = \frac{f}{m}.$$

4.7.1 Homogeneous Equations

A second order **homogeneous equation** is an 2nd order ODE for which r is zero, i.e. it is of the form

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0. \tag{9}$$

Notice that scalar multiples and sums of solutions are also solutions to this system. i.e. if y_1 and y_2 solve (9), then for any $a, b \in \mathbb{R}$

$$\begin{aligned}(ay_1 + by_2)'' + p(ay_1 + by_2)' + q(ay_1 + by_2) &= a(y_1'' + py_1' + qy_1) + b(y_2'' + py_2' + qy_2) \\ &= a \cdot 0 + b \cdot 0 = 0.\end{aligned}$$

Remembering back to our examples of vector space, we know that set of all functions is a vector space. Well, what we have just shown is that set of all solutions to (9) is a subspace of the set of all functions. i.e. the set of solutions

$$V :== \{y : [0 + \infty) \to \mathbb{R} \mid y''(t) + p(t)y'(t) + q(t)y(t) = 0\},$$

is a vector space. It is two dimensional.

That is to say, any solution y to (9) can be uniquely written as

$$y = ay_1 + by_2$$

where $a, b \in \mathbb{R}$ and y_1 and y_2 form a basis for V.

4.7.2 Inhomogeneous Equations

Suppose we have two solutions y_1 and y_2 to the inhomogeneous equation (8). Then $y_1 - y_2$ would solve (9). This means the following: If we can find one particular solution y_p to (8), one that contains no arbitrary constants, then all solutions will be of the form

$$y = y_p + y_h = y_p + ay_1 + by_2,$$

where $a, b \in \mathbb{R}$, and y_1, y_2 are the two basis functions for the homogeneous equation.

4.7.3 Solving a Problem

At this point we will assume p and q are real numbered constants in (8). We consider the equation

$$\ddot{x}(t) + b\dot{x}(t) + cx(t) = r(t), \tag{10}$$

where a and b are constants, and $r : \mathbb{R} \to \mathbb{R}$ is some given function.

How do we solve the homogeneous problem?

We begin by looking for solutions to the homogeneous problem ($r = 0$). We look for solutions of the form $x(t) = e^{\lambda t}$ where $\lambda \in \mathbb{C}$. Substituting this into the homogeneous ODE we have

$$\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0.$$

As $e^{\lambda t} \neq 0$ for any $t \in [0, +\infty)$, we have

$$p(\lambda) := \lambda^2 + b\lambda + c = 0.$$

This polynomial is called the **characteristic polynomial**, and $x(t)$ is a solution if λ is a root of p.

Thus there are three possibilities:

- There are 2 real roots $\lambda_1 \neq \lambda_2$. Then $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ form a basis for the solution space.
- There is 1 real root λ. Then $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ form a basis for the solution space.
- There are 2 complex roots $\lambda_\pm = \alpha \pm i\beta$. Then $y_1(t) = e^{\alpha t} \cos \beta t$ and $y_2(t) = e^{\alpha t} \sin \beta t$ form the basis for our solution case.

So for each for each these situations any solution to the homogeneous equation is of the form

$$y_h(t) = Ay_1(t) + By_2(t),$$

where A and B are constants, determined by the initial conditions.

How do we find a particular solution y_p?

We use the so called **method of undetermined coefficients**.

The idea: Look for a solution that looks like the right hand side function r of (10) with general coefficients. Determine these coefficients by substituting this into the problem. If the right hand side is of the same form as the solution to the homogeneous problem, multiply by t as in the case with a single root.

This method relies on you seeing examples and know what to look for in each case. It really is a method relying on intuition.

Example 4.28. We consider the equation

$$y'' + 2y' + 2y = -2e^{-t}\sin t. \tag{11}$$

The characteristic polynomial is

$$\lambda^2 + 2\lambda + 2 = 0$$

and so the roots are given by completing the square,

$$(\lambda + 1)^2 + 1 = 0.$$

So we have two roots $\lambda_\pm = -1 \pm i$. The basis for the homogeneous problem solution space is thus

$$y_1(t) = e^{-t}\cos t, \text{ and } y_2(t) = e^{-t}\sin t.$$

If the right hand side was not already part of the solution for the homogeneous problem, we would try for the particular solution

$$y_p(t) = Ae^{-t}\cos t + Be^{-t}\sin t$$

and determine A and B. Instead we will try

$$y_p(t) = Ate^{-t}\cos t + Bte^{-t}\sin t.$$

Substituting this into (11), we find that

$$2e^{-t}(B\cos(t) - A\sin(t)) = -2e^{-t}\sin t,$$

and so we find $B = 0$ and $A = 1$. Thus the general solution is

$$\begin{aligned} y(t) &= y_p(t) + C_1 y_1(t) + C_2 y_2(t) \\ &= te^{-t}\cos t + C_1 e^{-t}\cos t + C_2 e^{-t}\sin t, \end{aligned}$$

where C_1 and C_2 are arbitrary constants.

4.8 Linear Systems

In this section we will consider 1st order linear systems of the form

$$\begin{aligned}
\dot{x}_1 &= a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n, \\
\dot{x}_2 &= a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n, \\
&\vdots \\
\dot{x}_n &= a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n.
\end{aligned}$$

Or rather, setting

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

we are considering the problem

$$\dot{x} = Ax,$$

with

$$x(0) = x_0,$$

for some $x_0 \in \mathbb{R}$.

There a number of problems of this form. In fact, for non-linear systems $\dot{x} = f(x)$, the qualitative behaviour can be explored at equilibrium points x_0 by linearisation (to form a linear system like the above), and using the Hartman-Grobman theorem.

Example 4.29. In the fourth exercise set we introduced a population model between two locations, given by

$$\begin{aligned}
s_{n+1} &= 0.8s_n + 0.1f_n \\
f_{n+1} &= 0.2s_n + 0.9f_n.
\end{aligned}$$

These can be rearranged into

$$\begin{aligned}
s_{n+1} - s_n &= -0.2s_n + 0.1f_n \\
f_{n+1} - f_n &= 0.2s_n - 0.1f_n.
\end{aligned}$$

The continuous extension for this problem is

$$\dot{s} = -0.2s + 0.1f$$
$$\dot{f} = 0.2s - 0.1f,$$

or rather

$$\dot{x} = Ax, \text{ where } A = \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{bmatrix}.$$

Note that

$$\frac{d}{dt}(s+f) = \dot{s} + \dot{f} = -0.2s + 0.1f + 0.2s - 0.1f = 0,$$

so the total population between the two locations is preserved.

Example 4.30. The ODE

$$y'' + 2y' + 2y = 0$$

can be converted to a linear system by setting $x_1 = y'$ and $x_2 = y$, giving

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

4.8.1 Solving these Systems

In the one dimensional case, the IVP

$$\dot{x} = ax, \qquad x(0) = x_0,$$

where $a \in \mathbb{R}$, has a solution

$$x(t) = x_0 e^{at}.$$

Analogously, the solution of

$$\dot{x} = Ax, \qquad x(0) = x_0,$$

for $x_0 \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ has the solution

$$x(t) = e^{tA} x_0,$$

where here we have the matrix exponential.

We can check this using the definition of a solution. First, at 0,

$$x(0) = e^{0 \cdot A} x_0 = \left(\sum_{n=0}^{\infty} \frac{1}{n!} (0 \cdot A)^n \right) = I x_0 = x_0,$$

where we have used $0^0 = I$.

Now, taking the derivative

$$\begin{aligned}
\dot{x}(t) = (e^{tA})' = \left(\sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n x_0\right)' &= \left(\sum_{n=1}^{\infty} \frac{1}{(n-1)!} t^{n-1} A^n x_0\right) \\
&= A\left(\sum_{n=1}^{\infty} \frac{1}{(n-1)!} t^{n-1} A^{n-1} x_0\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n x_0\right) \\
&= A e^{tA} x_0 \\
&= A x(t).
\end{aligned}$$

We can write any matrix in Jordan normal form, i.e. for some matrix J,

$$A = PJP^{-1},$$

and

$$e^{tA} = e^{tPJP^{-1}} = \cdots = Pe^{tJ}P^{-1}.$$

If, even further A is diagonalisable, with D the diagonal matrix similar to A, then we know

$$P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \text{ and } D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

where the v_i are eigenvectors with associated eigenvalues λ_i. We also know that

$$e^{tA} = Pe^{tD}P^{-1}, \text{ where } e^{tD} = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}.$$

With this, we can construct a solution formula for $x(t) = Ax$ with $x(0) = x_0$. Begin by defining $a \in \mathbb{R}^n$ by $a = P^{-1}x_0$. Then

$$\begin{aligned}
x(t) = e^{tA}x_0 = Pe^{tD}P^{-1}x_0 = Pe^{tD}a \\
&= \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \\
&= a_1 e^{\lambda_1 t} v_1 + a_2 e^{\lambda_2 t} v_2 + \cdots + a_n e^{\lambda_n t} v_n
\end{aligned}$$

where $\lambda_i \in \mathbb{C}$ and $v_i \in \mathbb{C}^n$ for $i = 1, \ldots, n$.

What this tells us is that

$$\{e^{\lambda_1 t} v_1, e^{\lambda_2 t} v_2, \ldots, e^{\lambda_n t} v_n\}$$

forms a basis for the solution space.

4.8.2 Behaviour as $t \to \infty$

First note that

$$|e^{\lambda t}| = |e^{(\mathrm{Re}(\lambda)+\mathrm{Im}(\lambda))t}| = e^{\mathrm{Re}(\lambda)t},$$

so the basis functions behaviour at infinity is controlled by the eigenvalue. As the solution is a linear combination of these basis functions, its behaviour is determined by the eigenvectors and eigenvalues of the matrix A.

For the linear system $\dot{x} = Ax$ 0 is always an equilibrium point. Its behaviour is determined by the eigenvalues.

- Suppose for all eigenvalues λ_i, $\mathrm{Re}(\lambda_i) < 0$. Then all solutions will go to zero as $t \to \infty$, as $e^{\mathrm{Re}(\lambda_i)t} \to 0$ for all $i = 1, \ldots, n$. In this case 0 is an asymptotically stable equilibrium point.

- On the other hand, consider if for some λ_i that $\mathrm{Re}(\lambda_i) > 0$. Then there exists a solution x such that $|x(t) \to \infty$ as $t \to \infty$. This is because the basis function $e^{\mathrm{Re}(\lambda_i)t} \to \infty$. Thus 0 is an unstable equilibrium point.

- Finally, if $\mathrm{Re}(\lambda_i) = 0$ for all $i = 1, \ldots, n$, the solutions stay bounded, so if they don't start at zero they remain equidistant from zero for all time.

Note (Non-linear systems). Suppose we have a general non-linear system $\dot{x} = f(x)$ with x_0 as an equilibrium point. If the eigenvalues of the Jacobian $Df(x)$ all satisfy $\mathrm{Re}(\lambda_i) \neq 0$, then the "local" behaviour around the equilibrium point is the same as for the linearised system. This is due to the Hartman-Grobman theorem.

4.8.3 Notes and Examples

We begine by noting the following observation. If we were to make a change of variables $y = P^{-1}x$, then the system would become

$$\dot{y} = Dy,$$

which is a much easier system to solve.

Example 4.31. Consider the population model example we had previously,

$$A = \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{bmatrix} = PDP^{-1}$$

where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} -0.3 & 0 \\ 0 & 0 \end{bmatrix}.$$

Setting $a = P^{-1}x_0$, we have

$$x(t) = e^{tA}x_0 = Pe^{tD}P^{-1}x_0 = Pe^{tD}a = a_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.3t} + a_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Suppose we set $a_1 = 3$ and $a_2 = 4$, then

$$x(0) = 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

This tells us that the solution for $x_0 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ is

$$x(t) = \begin{bmatrix} 3 \\ -3 \end{bmatrix} e^{-0.3t} + \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

which goes to $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$ as $t \to \infty$.

Example 4.32. We consider again the problem

$$y'' + 2y' + 2y = 0.$$

As a system, with $x_1 = y'$, $x_2 = y$, we have $\dot{x} = Ax$ with

$$A = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_\pm = -1 \pm i$, with eigenvectors

$$v_\pm = \begin{bmatrix} -1 \pm i \\ 1 \end{bmatrix}.$$

Thus we can write A as

$$A = \begin{bmatrix} -1+i & -1-i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1+i & 0 \\ 0 & -1-i \end{bmatrix} P^{-1}.$$

The basis for the solution space is thus $\{e^{\lambda_+ t} v_+, e^{\lambda_- t} v_-\}$. Via Eulers formula,

$$e^{(-1\pm i)t} = e^{-t}(\cos t \pm i \sin t).$$

Hence the general solution is, for any arbitrary constants $a_1, a_2 \in \mathbb{R}$,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_1 e^{-t} \begin{bmatrix} -\cos t - i \sin t + i \cos t - \sin t \\ \cos t + i \sin t \end{bmatrix} + a_2 e^{-t} \begin{bmatrix} -\cos t + i \sin t + \sin t - i \cos t \\ \cos t - i \sin t \end{bmatrix}$$

We are interesting in the solution $y = x_2$ here. So focusing on the second row, we have the solution is

$$y(t) = e^{-t}((a_1 + a_2)\cos t + (a_1 - a_2) i \sin t).$$

These constants are arbitrary, so we can set new arbitrary constants $A, B \in \mathbb{R}$ by setting

$$a_1 = \frac{1}{2}(A + iB), \qquad a_2 = \frac{1}{2}(A - iB).$$

Then

$$a_1 + a_2 = A, \text{ and } a_1 - a_2 = iB,$$

and thus

$$y(t) = e^{-t}(A \cos t + B \sin t),$$

for arbitrary constants $A, B \in \mathbb{R}$, which reflects our previous solution.

Example 4.33 (A Modelling Example)**.** Consider two tanks with 100L volumes, each filled with salt water. The weight of the salt water at time $t \in [0, +\infty)$ in the respective tanks is $x_1(t)$ and $x_2(t)$.

Suppose the tanks are connected via a pair of pipes, and the flow rate between the two tanks is $g = 1\text{Ls}^{-1}$ each way.

The rate of change of salinity is thus given by

$$\dot{x}_1 = -\frac{x_1}{V} g + \frac{x_2}{g} V$$
$$\dot{x}_2 = \frac{x_1}{V} g - \frac{x_2}{g} V.$$

This thus gives us a system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \frac{1}{100} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

4.8.4 Inhomogeneous Equations

To finish off, we consider the problem of the form

$$\dot{x} = Ax + f, \qquad x(0) = x_0,$$

where A is some $n \times n$ matrix, and $f : \mathbb{R} \to \mathbb{R}^n$ is some given function, called a source term.

This problem resembles the form of the problem for the integration factor method we discussed earlier. Inspired by this, we multiply by e^{-tA}, and perform some algebraic manipulation

$$e^{-tA}\dot{x}(t) - Ae^{-tA}x(t) = e^{-tA}f(t),$$

or rather

$$\frac{d}{dt}\left(e^{-tA}x(t)\right) = e^{-tA}f(t).$$

Thus, after integrating from 0 to t, we have

$$e^{-tA}x(t) - x(0) = \int_0^t e^{-sA}f(s)\ ds,$$

and thus

$$x(t) = e^{-tA}x(0) + \int_0^t e^{(t-s)A}f(s)ds.$$

Notice how the first term solves the homogeneous problem

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0.$$