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EXAM IN TMA4115 MATHEMATICS 3  
English  
Monday 4th June 2012  
Time: 9-13

Examination Aids (code C): Simple calculator (HP30S or Citizen SR-270X)  
Rottman: *Matematisk formelsamling*

Sensur: 25th June 2012

Give reasons for all answers, ensuring that it is clear how the answer has been reached. Each of the 12 parts (1,2a,2b,3,4,5a,5b,6,7a,7b,8a,8b) has the same weight.

**Problem 1** Solve  $w^2 = (-1 + i\sqrt{3})/2$ .

Find all solutions of the equation  $z^4 + z^2 + 1 = 0$  and draw them in the complex plane. Write the solutions in the form  $x + iy$ .

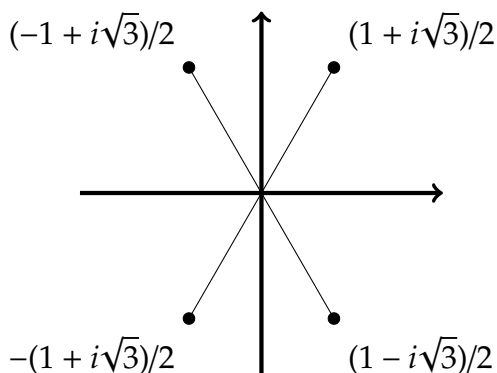
**Solution:**

Write  $(-1 + i\sqrt{3})/2$  in polar form: its length is  $\sqrt{(-1/2)^2 + (\sqrt{3}/2)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$  and its argument is  $\tan^{-1}(-\sqrt{3}) = 2\pi/3$ . Thus  $(-1 + i\sqrt{3})/2 = e^{i2\pi/3}$ . Its square roots are thus  $\pm e^{i\pi/3}$ . In cartesian form, these are  $\pm(1 + i\sqrt{3})/2$

To solve  $z^4 + z^2 + 1 = 0$  we start by noticing that this is a quadratic in  $z^2$ . Using the quadratic formula, we find that:

$$z^2 = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2} = (-1 \pm i\sqrt{3})/2.$$

From the first part, we get the two solutions  $\pm(1 + i\sqrt{3})/2$ . The quickest way to find the other two is to notice that  $\bar{z}^2 = \overline{z^2}$ , whence the other two solutions are  $\pm(1 - i\sqrt{3})/2$ .



### Problem 2

- a) Find a particular solution of  $y'' - 4y' + y = te^t + t$ .

**Solution:**

The term  $te^t$  looks the more difficult to achieve so we begin with that. Let us try  $y_1 = te^t$ :

$$y_1'' - 4y_1' + y_1 = (2e^t + te^t) - 4(e^t + te^t) + te^t = -2te^t - 2e^t.$$

Hence  $y_2 = -\frac{1}{2}te^t$  produces the desired  $te^t$  but also introduces a term of  $e^t$ . To correct for that, we try  $y_3 = e^t$ :

$$y_3'' - 4y_3' + y_3 = e^t - 4e^t + e^t = -2e^t.$$

Thus  $y_4 = -\frac{1}{2}te^t + \frac{1}{2}e^t$  produces  $te^t$ .

To get the  $t$ , we try  $y_5 = t$ :

$$y_5'' - 4y_5' + y_5 = 0 - 4 \cdot 1 + t = t - 4.$$

To correct for the  $-4$ , we try  $y_6 = 4$ :

$$y_6'' - 4y_6' + y_6 = 0 - 4 \cdot 0 + 4 = 4.$$

Hence  $y_7 = t + 4$  produces  $t$ .

Thus a particular solution is:

$$-\frac{1}{2}te^t + \frac{1}{2}e^t + t + 4.$$

Quick check:

$y''$	:	$-\frac{1}{2}te^t$	$-e^t + \frac{1}{2}e^t$	
$-4y'$	:	$2te^t$	$+2e^t - 2e^t$	$-4$
$+y$	:	$-\frac{1}{2}te^t$	$+\frac{1}{2}e^t$	$+t + 4$
		$te^t$	$+t$	

b) Find the solution of  $y'' - 4y' + y = te^t + t$ , where  $y'(0) = y(0) = 0$ .

**Solution:**

We have a particular solution, what remains is to find the solution to the homogeneous equation and to fit the resulting general solution to the initial conditions.

The solutions to the homogeneous equation will be of the form  $Ae^{\alpha t} + Be^{\beta t}$  for some  $\alpha, \beta \in \mathbb{C}$ . These are the roots of the auxiliary equation  $s^2 - 4s + 1 = 0$ . Solving this via the quadratic formula we obtain:

$$\frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}.$$

Thus the general solution is:

$$Ae^{(2+\sqrt{3})t} + Be^{(2-\sqrt{3})t} - \frac{1}{2}te^t + \frac{1}{2}e^t + t + 4.$$

We now fit this to the initial conditions: at  $t = 0$ , the above evaluates to  $A+B+\frac{1}{2}+4$ , whence  $A + B = -\frac{9}{2}$ . Differentiating and evaluating, we obtain  $(2 + \sqrt{3})A + (2 - \sqrt{3})B - \frac{1}{2} + \frac{1}{2} + 1$ , whence  $(2 + \sqrt{3})A + (2 - \sqrt{3})B = -1$  or  $A - B = \frac{8}{\sqrt{3}}$ . Hence:

$$A = \frac{16 - 9\sqrt{3}}{4\sqrt{3}}, \quad B = \frac{-16 - 9\sqrt{3}}{4\sqrt{3}}$$

and thus the solution is:

$$\frac{16 - 9\sqrt{3}}{4\sqrt{3}}e^{(2+\sqrt{3})t} + \frac{-16 - 9\sqrt{3}}{4\sqrt{3}}e^{(2-\sqrt{3})t} - \frac{1}{2}te^t + \frac{1}{2}e^t + t + 4.$$



**Problem 3** Let  $a$  be a real number. Find the general solution of  $y'' + ay = \cos x$ .

**Solution:**

The solution of the homogeneous equation will depend on  $a$ . Let  $\lambda = \sqrt{|a|}$ , then:

- $a < 0$ :  $Ae^{\lambda x} + Be^{-\lambda x}$ ,
- $a = 0$ :  $Ax + B$ ,
- $a > 0$ :  $A \cos(\lambda x) + B \sin(\lambda x)$

The particular solution will depend on whether or not  $a = 1$ . If  $a \neq 1$ , then a particular solution is  $\frac{1}{a-1} \cos(x)$ . If  $a = 1$  then  $\cos(x)$  is a solution of the homogeneous equation and so we expect a solution of the form  $Ax \cos(x) + Bx \sin(x)$ . Substituting in, we find that if  $y_1(x) = x \cos(x)$  then:

$$y_1'' + y_1 = -x \cos(x) - \sin(x) - \sin(x) + x \cos(x) = -2 \sin(x)$$

whilst if  $y_1(x) = x \sin(x)$  then:

$$y_2'' + y_2 = -x \sin(x) + 2 \cos(x) + x \sin(x) = 2 \cos(x)$$

hence the general solution is:

- $a < 0$ :  $Ae^{\lambda x} + Be^{-\lambda x} + \frac{1}{a-1} \cos(x)$ ,
- $a = 0$ :  $Ax + B - \cos(x)$ ,
- $a > 0, a \neq 1$ :  $A \cos(\lambda x) + B \sin(\lambda x) + \frac{1}{a-1} \cos(x)$ ,
- $a = 1$ :  $A \cos(x) + B \sin(x) + \frac{1}{2} x \sin(x)$



**Problem 4** Find the least squares solution of

$$\begin{cases} -y + z = 1 \\ 2y - z = 0 \\ x + y + 3z = 0 \\ -x + 3y + z = 0 \end{cases}$$

**Solution:**

This is equivalent to the following matrix equation:

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let  $A$  be the matrix and  $b$  the target vector. For a least-squares solution, we solve  $A^T Ax = A^T b$  so we compute:

$$A^T A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 2 & 1 & 3 \\ 1 & -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ -2 & 15 & 3 \\ 2 & 3 & 12 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 2 & 1 & 3 \\ 1 & -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

We solve the resulting linear system:

$$\begin{aligned} \begin{bmatrix} 2 & -2 & 2 & 0 \\ -2 & 15 & 3 & -1 \\ 2 & 3 & 12 & 1 \end{bmatrix} &\mapsto \begin{bmatrix} 2 & -2 & 2 & 0 \\ 0 & 13 & 5 & -1 \\ 0 & 5 & 10 & 1 \end{bmatrix} &\mapsto \begin{bmatrix} 2 & -2 & 2 & 0 \\ 0 & 13 & 5 & -1 \\ 0 & 0 & 105 & 18 \end{bmatrix} \\ \mapsto \begin{bmatrix} 2 & -2 & 2 & 0 \\ 0 & 13 & 5 & -1 \\ 0 & 0 & 35 & 6 \end{bmatrix} &\mapsto \begin{bmatrix} 2 & -2 & 2 & 0 \\ 0 & 7 & 0 & -1 \\ 0 & 0 & 35 & 6 \end{bmatrix} &\mapsto \begin{bmatrix} 35 & 0 & 0 & -11 \\ 0 & 7 & 0 & -1 \\ 0 & 0 & 35 & 6 \end{bmatrix} \end{aligned}$$

Whence the least-squares solution is:

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} -11/35 \\ -1/7 \\ 6/35 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 11 \\ -16 \\ 2 \\ 2 \end{bmatrix}$$



**Problem 5** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be defined by  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y + z \\ -x + 3y + z \\ 2x - z \\ y + 4z \end{bmatrix}$ .

a) Find a matrix  $A$  such that  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ .

**Solution:**

The matrix is:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$



b) Find  $\dim \text{Null}(A)$  and a basis for  $\text{Col}(A)$ . Is  $T$  one-to-one (injective)? Is  $T$  onto (surjective)?

**Solution:**

We row reduce the matrix as follows:

$$\begin{aligned} \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix} &\mapsto \begin{bmatrix} 1 & -3 & -1 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix} &\mapsto \begin{bmatrix} 1 & -3 & 1 \\ 0 & 7 & 3 \\ 0 & 6 & 1 \\ 0 & 1 & 4 \end{bmatrix} \\ &\mapsto \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 4 \\ 0 & 7 & 3 \\ 0 & 6 & 1 \end{bmatrix} &\mapsto \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -23 \\ 0 & 0 & -25 \end{bmatrix} &\mapsto \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Hence  $\dim \text{Null}(A) = 0$  and a basis for  $\text{Col}(A)$  is the columns of  $A$ :

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 4 \end{bmatrix} \right\}$$

As  $\dim \text{Null}(A) = 0$ ,  $T$  is injective. As  $\dim \text{Col}(A) = 3$ ,  $T$  is not surjective.



**Problem 6** Let  $A$  be a  $4 \times 4$  matrix. Let  $B = \begin{bmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ .

Assume that  $\det(AB) = 4$ . What is  $\det(A)$ ?

**Solution:**

As  $\det(AB) = \det(A) \det(B)$ , we have that  $\det(A) = 4/\det(B)$ . We compute  $\det(B)$  as:

$$\begin{vmatrix} 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -2.$$

Hence  $\det(A) = -2$ .



Show that the equation  $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$  has only the solution  $x_1 = x_2 = x_3 = x_4 = 0$ .

**Solution:**

As  $\det(A) \neq 0$ ,  $A$  is an invertible matrix. Hence it is injective, and so the only solution of  $Ax = 0$  is  $x = 0$ .



**Problem 7**

a) Find all the eigenvalues of  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix}$ .

**Solution:**

There is an obvious eigenvalue, 2, with eigenvector  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ . Another eigenvalue can be

seen from the fact that the “row sums” of the lower two rows are both 3, hence  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

is an eigenvector with eigenvalue 3. To find the last eigenvalue, we can use the fact that the sum of the roots of the characteristic polynomial (with multiplicity) is the same as the sum of the diagonal terms, which is 7, whence the last root is again 2. Hence the eigenvalues are 2 and 3.

In the more traditional fashion, we could compute these as follows. We start by computing the characteristic polynomial:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & -2 \\ 0 & 1 & \lambda - 4 \end{vmatrix} = (\lambda - 2)((\lambda - 1)(\lambda - 4) + 2) = (\lambda - 2)(\lambda^2 - 5\lambda + 6).$$

This has an obvious root,  $\lambda = 2$ . The other roots are roots of the quadratic factor, which we can either solve using the quadratic formula or simply see that the roots are 2 and 3. Hence the roots of the characteristic polynomial are 2, 2, 3, whence the eigenvalues are 2 and 3.



b) Find a basis for each eigenspace of  $A$ . Is  $A$  diagonalisable?

**Solution:**

As remarked above, two eigenvectors are  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$  for eigenvalues 2 and 3 respectively. As the repeated root was 2, if there is a third eigenvector (linearly independent of these two) it will have to have eigenvalue 2. Thus we compute the null space of  $2I - A$ :

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Thus a basis for the eigenspace corresponding to eigenvalue 2 is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

and for the eigenspace corresponding to eigenvalue 3 is:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

The resulting three vectors are linearly independent, whence form a basis for  $\mathbb{R}^3$ . Thus there is a basis of  $\mathbb{R}^3$  consisting of eigenvectors of  $A$  and so  $A$  is diagonalisable.



**Problem 8** Let  $A = \begin{bmatrix} -2 & -5 \\ 5 & -2 \end{bmatrix}$ .

a) Find the complex eigenvalues of  $A$  and the corresponding eigenvectors in  $\mathbb{C}^2$ .

**Solution:**

We compute  $\det(\lambda I - A)$ :

$$\begin{vmatrix} \lambda + 2 & 5 \\ -5 & \lambda + 2 \end{vmatrix} = (\lambda + 2)^2 + 25 = \lambda^2 + 4\lambda + 29.$$

Using the quadratic formula, we find the roots are given by:

$$\frac{-4 \pm \sqrt{16 - 4 \cdot 29}}{2} = -2 \pm \sqrt{-25} = -2 \pm 5i.$$



To find the eigenvectors, we find the corresponding null spaces. Starting with  $-2+5i$ , we compute:

$$\begin{bmatrix} 5i & 5 \\ -5 & 5i \end{bmatrix} \mapsto \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \mapsto \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

As  $A$  has only real values, its eigenvalues and eigenvectors come in conjugate pairs so the other eigenvector (with eigenvalue  $-2-5i$ ) is  $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ .



b) Find the solution of the system of differential equations  $\vec{y}'(t) = A\vec{y}(t)$  satisfying  $\vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

The answer should be written in the form  $\vec{y}(t) = e^{\lambda t} \begin{bmatrix} a \cos(\omega t) + b \sin(\omega t) \\ c \cos(\omega t) + d \sin(\omega t) \end{bmatrix}$ .

**Solution:**

The general solution is:

$$y(t) = \alpha e^{\lambda_1 t} \begin{bmatrix} i \\ 1 \end{bmatrix} + \beta e^{\lambda_2 t} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

with  $\alpha, \beta \in \mathbb{C}$ ,  $\lambda_1 = -2 + 5i$ , and  $\lambda_2 = -2 - 5i$ .

The condition at  $t = 0$  is that:

$$\alpha \begin{bmatrix} i \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

whence  $i(\alpha - \beta) = 1$  and  $\alpha + \beta = 0$ . Hence  $\beta = i/2$  and  $\alpha = -i/2$ . Thus the solution is:

$$\begin{aligned} y(t) &= -\frac{1}{2}ie^{\lambda_1 t} \begin{bmatrix} i \\ 1 \end{bmatrix} + \frac{1}{2}ie^{\lambda_2 t} \begin{bmatrix} -i \\ 1 \end{bmatrix} \\ &= e^{-2t} \begin{bmatrix} \frac{1}{2}e^{5it} + \frac{1}{2}e^{-5it} \\ -i\frac{1}{2}e^{5it} + i\frac{1}{2}e^{-5it} \end{bmatrix} \\ &= e^{-2t} \begin{bmatrix} \cos(5t) \\ \sin(5t) \end{bmatrix} \end{aligned}$$

