Norwegian University of Science and Technology
Institutt for matematiske fag

Solutions to exercise set 3

Exercises from the textbook, s. lv

2 The characteristic equation is

$$2\lambda^2 - 3\lambda - 2 = 0.$$

It has two real roots $\lambda_{1,2} = (3 \pm 5)/4$, $\lambda_1 = 2$, $\lambda_2 = -1/2$. Then the general solution is given by

$$y(t) = C_1 e^{2t} + C_2 e^{-t/2}.$$

14 The characteristic equation is

$$\lambda^2 + 2\lambda + 3 = 0.$$

It has two complex conjugate roots $\lambda_{1,2} = (-2 \pm \sqrt{-8})/2$, $\lambda_1 = -1 + i\sqrt{2}$, $\lambda_2 = -1 - i\sqrt{2}$. Then the general solution is given by

$$y(t) = e^{-t}(C_1 \cos(\sqrt{2}t) + C_2 \sin(\sqrt{2}t)).$$

19 The characteristic equation is

$$4\lambda^2 + 4\lambda + 1 = 0.$$

It has a double root $\lambda = -1/2$. Then one solution is $y_1(t) = e^{-t/2}$ and the second solution is $y_2(t) = te^{-t/2}$. The general solution is

$$y(t) = e^{-t/2}(C_1 + C_2 t).$$

27 First, we find the genaral solution. The characteristic equation

$$\lambda^2 - 2\lambda + 17 = 0$$

has two complex conjugate roots, $\lambda_{1,2} = 1 \pm 4i$. Then the general solution is

$$y(t) = e^t(C_1 \cos 4t + C_2 \sin 4t).$$

We use initial data to determine the values of C_1 and C_2. We have

$$-2 = y(0) = e^0(C_1 \cos 0 + C_2 \sin 0) = C_1$$

and

$$y'(t) = e^t(C_1 \cos 4t + C_2 \sin 4t - 4C_1 \sin 4t + 4C_2 \cos 4t)$$
$$3 = y'(0) = C_1 + 4C_2 = 4C_2 - 2$$
$$C_2 = \frac{5}{4}$$

the solution is

$$y(t) = e^t(-2\cos 4t + \frac{5}{4}\sin 4t).$$

Exercises from the textbook, s. lxii-lxiii

11 Substitute $m = 0.2$ kg, $\mu = 0$ kg/s, and $k = 5$ kg/s^2 in the equation $mx'' + \mu x' + kx = 0$, we obtain $0.2y'' + 5y = 0$ or

$$y'' + 25y = 0.$$

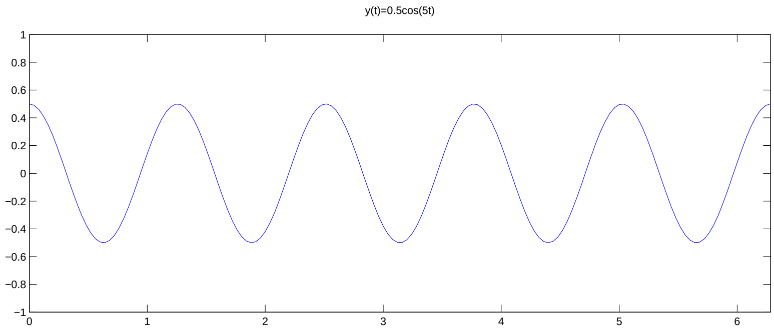
The characteristic equation is $\lambda^2 + 25 = 0$, with zeros $\lambda_{1,2} = \pm 5i$. The general solution is

$$y(t) = C_1 \cos 5t + C_2 \sin 5t.$$

The initial displacement is $y(0) = 0.5$ m and the initial velocity is $y'(0) = 0$ m/s. We have $y(0) = C_1$ and $y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t$, $y'(0) = 5C_2$. Then $C_1 = 0.5$, $C_2 = 0$. Thus the solution is

$$y(t) = 0.5 \cos 5t.$$

The amlitude is 0.5 m, the frequancy is 5 rad/s and the fase is $\varphi = 0$ rad.



16 a) By Hooke's Law,

$$mg = kl \Rightarrow k = \frac{mg}{l} = \frac{1 \cdot 9.8}{4.9} = 2$$

$k = 2$ kg/s^2.

b) The equation is

$$my'' + \mu y' + ky = 0,$$

with $m = 1$ kg, $\mu = 3$ kg/s and $k = 2$ kg/s^2 it becomes

$$y'' + 3y' + 2y = 0.$$

Then the characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$, and the roots are $\lambda_1 = -1$ and $\lambda_2 = -2$. The general solution is then

$$y(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

The initial data are $y(0) = 1$ m and $y'(0) = 1$ m/s, hence

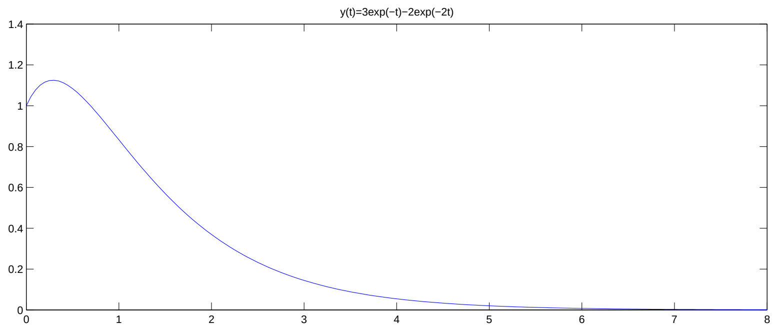
$$-1 = y(0) = C_1 + C_2$$

$$y'(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$$

$$-1 = y'(0) = -C_1 - 2C_2$$

We solve the equations and get $C_1 = 3$ og $C_2 = -2$. Then the position as function of time is given by (with positive direction downward)

$$y(t) = 3e^{-t} - 2e^{-2t}.$$



24 First, we find the spring constant

$$mg = kl \Rightarrow k = \frac{mg}{l} = \frac{10 \cdot 9.8}{1} = 98,$$

$k = 98$ kg/s^2.

Then the equation becomes

$$10y'' + 20y' + 98y = 0$$

$$y'' + 2y' + 9.8y = 0.$$

Thus the characteristic equation is $\lambda^2 + 2\lambda + 9.8 = 0$, with two complex roots $\lambda_{1,2} = -1 \pm \omega_0 i$, where $\omega_0 \approx 2.9665$. The general solution is

$$y(t) = e^{-t}(C_1 \cos \omega_0 t + C_2 \sin \omega_0 t).$$

The initial data is $y(0) = 0$ m and $y'(0) = 1.2$ m/s. We determine C_1 and C_2,

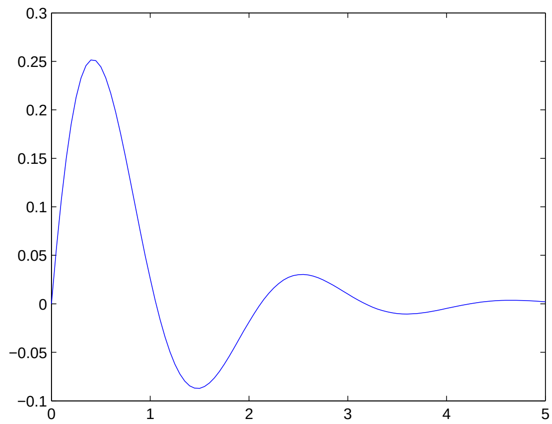
$$0 = y(0) = C_1$$

$$y'(t) = e^{-t}(-C_1 \cos \omega_0 t - C_2 \sin \omega_0 t - \omega_0 C_1 \sin \omega_0 t + \omega_0 C_2 \cos \omega_0 t)$$
$$1.2 = y'(0) = -C_1 + \omega_0 C_2$$
$$C_2 \approx 0.4045$$

Then the solution is

$$y(t) = C_2 e^{-t} \sin \omega_0 t = C_2 e^{-t} \cos(\omega_0 t - \frac{\pi}{2}).$$

The amplitude is $C_2 \approx 0.4045$, the frequency is $\omega_0 \approx 2.9665$ and the fase is $\frac{\pi}{2}$.



Exercises from the textbook, s. lxxi-lxxii

3 We look for a particular solution of the form $y_p(t) = ae^{-t}$, where a is an undetermined coefficient. The derivatives of y_p are $y_p' = -ae^{-t}$ and $y_p'' = ae^{-t}$. Inserting this into the equation, we get

$$ae^{-t} + 2(-ae^{-t}) + 5ae^{-t} = 12e^{-t}.$$

Hence $4a = 12$ and $a = 3$. The solution is $y_p(t) = 3e^{-t}$.

8 We are looking for a particular solution of the form $y_p(t) = a\cos 3t + b\sin 3t$. The derivates are

$$y_p' = -3a\sin 3t + 3b\cos 3t,$$
$$y_p'' = -9a\cos 3t - 9b\sin 3t.$$

We insert these expressions into the equation and get

$$-9a\cos 3t - 9b\sin 3t + 7(-3a\sin 3t + 3b\cos 3t) + 10(a\cos 3t + b\sin 3t) = -4\sin 3t.$$

Comparing the coefficients of the sine and cosine terms, we obtain $-9a+21b+10a = 0$ and $-9b - 21a + 10b = -4$. The solution is $a = \frac{42}{221}$ and $b = -\frac{2}{221}$. The particular solution is

$$y_p = \frac{42}{221}\cos 3t - \frac{2}{221}\sin 3t.$$

12 We first consider the complex equation

$$z'' + 7z' + 6z = 3e^{2it},$$

and look for a particular solution of the form $z_p = ae^{2it}$. Then $z_p' = 2iae^{2it}$ and $z_p'' = -4ae^{-2it}$. We insert these in the equation and get

$$(-4a + 14ia + 6a)e^{2it} = 3e^{2it}$$

$$2a + 14ia = 3$$

$$a = \frac{3}{2+14i} = \frac{3}{100} - i\frac{21}{100}$$

Hence

$$z_p = \frac{3-21i}{100}e^{2it} = \frac{3}{100}\cos 2t + \frac{21}{100}\sin 2t + i(\frac{3}{100}\sin 2t - \frac{21}{100}\cos 2t).$$

The imaginary part of this solution is a particular solution of the equation $y'' + 7y' + 6y = 3\sin 2t$,

$$y_p = \frac{3}{100}\sin 2t - \frac{21}{100}\cos 2t.$$

21 First we solve the homogeneous equation

$$y'' - 2y' + 5y = 0.$$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$, it has roots $\lambda_{1,2} = 1 \pm 2i$. Then the general solution of the homogeneous equation is

$$y_h = e^t(C_1\cos 2t + C_2\sin 2t).$$

We look for a particular solution of the form $y = A_1\cos t + A_2\sin t$ and compute the derivatives $y' = -A_1\sin t + A_2\cos t$ and $y'' = -A_1\cos t - A_2\sin t$. Inserting them into the equation, we get

$$-A_1\cos t - A_2\sin t - 2(-A_1\sin t + A_2\cos t) + 5(A_1\cos t + A_2\sin t) = 3\cos t.$$

Then

$$-A_1\cos t - 2A_2\cos t + 5A_1\cos t = 3\cos t$$
$$-A_2\sin t + 2A_1\sin t + 5A_2\sin t = 0.$$

We have $4A_1 - 2A_2 = 3$ and $2A_1 + 4A_2 = 0$. Solving the system, we obtain $A_1 = \frac{3}{5}$ og $A_2 = -\frac{3}{10}$, hence a particular solution is

$$y_p = \frac{3}{5}\cos t - \frac{3}{10}\sin t.$$

and the general solution is

$$y = e^t(C_1\cos 2t + C_2\sin 2t) + \frac{3}{5}\cos t - \frac{3}{10}\sin t.$$

Finally, we solve the initial value problem,

$$0 = y(0) = e^0(C_1\cos 0 + C_2\sin 0) + \frac{3}{5}\cos t - \frac{3}{10}\sin 0$$

$$y'(t) = e^t(C_1\cos 2t + C_2\sin 2t) + e^t(-2C_1\sin 2t + 2C_2\cos 2t) - \frac{3}{5}\sin t - \frac{3}{10}\cos t$$

$$-2 = y'(0) = C_1 + 2C_2 - \frac{3}{10}$$

Therefore $C_1 = -\frac{3}{5}$ and $C_2 = \frac{1}{2}(-2 - C_1 + \frac{3}{10}) = -\frac{11}{20}$.

The answer is

$$y = e^t(-\frac{3}{5}\cos 2t - \frac{11}{20}\sin 2t) + \frac{3}{5}\cos t - \frac{3}{10}\sin t.$$

27 We see that the forcing term $\sin 3t$ is a solution of the homogeneous equation $y''+9y = 0$. We then look for a particular solution of the form $y_p(t) = t(a\cos 3t + b\sin 3t)$. We compute the derivatives:

$$y_p' = a\cos 3t + b\sin 3t + t(-3a\sin 3t + 3b\cos 3t),$$

$$y_p'' = 2(-3a\sin t + 3b\cos 3t) + t(-9a\cos 3t - 9b\sin 3t).$$

Substitute the derivatives into the inhomogeneous equation and get

$$2(-3a\sin 3t + 3b\cos 3t) - 9t(a\cos 3t + b\sin 3t) + 9t(a\cos 3t + b\sin 3t) = \sin 3t.$$

We obtain $a = -1/6$, $b = 0$ and $y_p = -t/6\cos 3t$.

An alternative solution: We can also use the complex method and solve the equation

$$z'' + 9z = e^{3it}.$$

We see that $z = e^{3it}$ is a solution of the homogeneous equation, so we try $z_p = ate^{3it}$. We get $z_p' = ae^{3it} + 3iate^{3it}$ and $z_p'' = 6iae^{3it} - 9ate^{3it}$ and substitute into the equation

$$6iae^{3it} - 9ate^{3it} + 9ate^{3it} = e^{3it}.$$

Hence, $6ia = 1$ and $a = -i/6$. Then $z_p = -it/6(\cos 3t + i\sin 3t)$, we take the imaginary part of the complex solution and obtain $y_p = -t/6\cos 3t$.

Exam problem

2 **a)** The characteristic eaution is $\lambda^2 - \lambda - 2 = 0$, it has two roots $\lambda_1 = 2$ and $\lambda_2 = -1$. The general solution of the homogeneous equation is $y(x) = C_1e^{2x} + C_2e^{-x}$. We determine the values of C_1 and C_2 from the initial data $y(0) = 3$ and $y'(0) = 0$. We have $3 = y(0) = C_1 + C_2$, $y'(x) = 2C_1e^{2x} - C_2e^{-x}$ and $0 = y'(0) = 2C_1 - C_2$. Then $C_1 = 1$ and $C_2 = 2$. The solution of the initial value problem is $y(x) = e^{2x} + 2e^{-x}$.

b) The general solution of the inhomogeneous equation is $y(x) = y_h(x) + y_p(x)$, where y_h is the general solution of the corresponding homogeneous equation (it was obtained in a), $y_h = C_1e^{2x} + C_2e^{-x}$ and y_p is a particular solution. We use the method of undetermined coefficients and look for a particular solution og the form $y_p = A\cos x + B\sin x + Cxe^{2x}$ since e^{2x} is a solution of the homogeneous equation. We have

$$y_p' = -A\sin x + B\cos x + Ce^{2x} + 2Cxe^{2x},$$

$$y_p'' = -A\cos x - B\sin x + 4Ce^{2x} + 4Cxe^{2x}.$$

Inserting these into the equation gives

$$(-3A - B)\cos x + (-3B + A)\sin x + 3Ce^{2x} = 8\sin x + 3e^{2x}.$$

Equating the coefficients, we obtain

$$-3A - B = 0, \quad -3B + A = 8, \quad 3C = 3.$$

Then $A = 0.8, B = -2.4, C = 1$. Finally, $y_p = 0.8\cos x - 2.4\sin x + xe^{2x}$ and the general solution is

$$y(x) = C_1e^{2x} + C - 2e^{-x} + 0.8\cos x - 2.4\sin x + xe^{2x}.$$