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TMA4110 - Calculus 3

Lecture 5

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Reference group

We need a reference group (referansegruppe) for TMA4110. Send an email to `tokemeie@math.ntnu.no` if you are interested in participating in the reference group.

Second-order homogeneous linear differential equations

A second-order homogeneous linear differential equation is a differential equation with can be written on the form

$$y'' + p(t)y' + q(t)y = 0.$$

Theorem 1.23

Suppose that y_1 and y_2 are linearly independent solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

on the interval (a, b). Then any solution to the equation is a linear combination of y_1 and y_2.

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Example

Let us show that $y(t) = c_1 t^2 + c_2 t$ is the general solution of the second-order differential equation $y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 0$ on the interval $(0, \infty)$.

Let $y_1(t) = t^2$ and $y_2(t) = t$. $y_1'(t) = 2t$, $y_1''(t) = 2$ and $y_1'' - \frac{2}{t}y_1' + \frac{2}{t^2}y_1 = 2 - 4 + 2 = 0$. $y_2'(t) = 1$, $y_2''(t) = 0$ and $y_2'' - \frac{2}{t}y_2' + \frac{2}{t^2}y_2 = -\frac{2}{t} + \frac{2}{t} = 0$. So y_1 and y_2 are solutions. Suppose $c_1 y_1(t) + c_2 y_2(t) = 0$ for all t in $(0,\infty)$. By evaluating at $t=1$ and $t=2$ we see that $c_1 + c_2 = 0$ and $4c_1 + 2c_2 = 0$. $2c_1 = 4c_1 + 2c_2 - 2(c_1 + c_2) = 0$ so it follows that $c_1 = c_2 = 0$. Thus y_1 and y_2 are linearly independt on $(0,\infty)$ and the general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 t^2 + c_2 t$.

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Fundamental set of solutions

- Two linearly independent solutions to a second-order homogeneous linear differential equation is said to form a *fundamental set of solutions*.
- The previous result then says that if y_1, y_2 form a fundamental set of solutions to a second-order homogeneous linear differential equation, then any solution to that differential equation can be written as a linear combination of y_1 and y_2.
- If y_1 and y_2 are solution to a second-order homogeneous linear differential equation, then we can check if they form a fundamental set of solutions either
  1. by showing that neither is a constant multiple of the other,
  2. or by showing that the Wronskian of y_1 and y_2 is not zero at any point.

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Exercise 4.1.25

Show that $y_1(t) = e^{-4t}$ and $y_2(t) = te^{-4t}$ form a fundamental set of solutions for $y'' + 8y' + 16y = 0$, then find a solution satisfying $y(0) = 2$ and $y'(0) = -1$.

$y_1'(t) = -4e^{-4t}$, $y_1''(t) = 16e^{-4t}$ so $y_1''(t) + 8y_1'(t) + 16y_1(t) = 16e^{-4t} - 32e^{-4t} + 16e^{-4t} = 0$.

$y_2'(t) = e^{-4t} - 4te^{-4t}$, $y_2''(t) = -4e^{-4t} - 4e^{-4t} + 16te^{-4t}$ so

$y_2''(t) + 8y_2'(t) + 16y_2(t) = -8e^{-4t} + 16te^{-4t} + 8e^{-4t} - 32te^{-4t} + 16te^{-4t} = 0$,

so y_1 and y_2 are solutions.

$W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = e^{-4t}(e^{-4t} - 4te^{-4t}) - te^{-4t}(-4e^{-4t})$

$= e^{-8t} \neq 0$ so y_1 and y_2 form a fundamental set of solutions.

Exercise 4.1.25

Show that $y_1(t) = e^{-4t}$ and $y_2(t) = te^{-4t}$ form a fundamental set of solutions for $y'' + 8y' + 16y = 0$, then find a solution satisfying $y(0) = 2$ and $y'(0) = -1$.

Let $y(t) = c_1 e^{-4t} + c_2 te^{-4t}$. Then $y(0) = c_1$ and $y'(0) = c_1 y_1'(0) + c_2 y_2'(0) = -4c_1 + c_2$. So $y(0) = 2$ and $y'(0) = -1$ if and only if $c_1 = 2$ and $c_2 = 7$. So $y(t) = 2e^{-4t} + 7te^{-4t}$ is the solution we are looking for.

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Second-order homogeneous linear differential equations with constant coefficients

A second-order homogeneous linear differential equation with constant coefficients is a differential equation with can be written on the form

$$y'' + py' + qy = 0$$

where p and q are constants.

Finding solutions to second-order homogeneous linear differential equations with constant coefficients

Let

$$y'' + py' + qy = 0$$

be a second-order homogeneous linear differential equation with constant coefficients.
We will look for a solution of the type $y(t) = e^{\lambda t}$ where λ is a constant.

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Finding solutions to second-order homogeneous linear differential equations with constant coefficients

Inserting $y(t) = e^{\lambda t}$ into the differential equation, we obtain

$$y'' + py' + qy = \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} = (\lambda^2 + p\lambda + q)e^{\lambda t}.$$

Since $e^{\lambda t} \neq 0$, we have that $y(t) = e^{\lambda t}$ is a solution if and only if $\lambda^2 + p\lambda + q = 0$.

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Characteristic polynomial

The polynomial $\lambda^2 + p\lambda + q$ is called the *characteristic polynomial* of the differential equation $y'' + py' + qy = 0$.
A root of the characteristic polynomial is called a *characteristic root*.
The characteristic roots are

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Looking at the discriminant $p^2 - 4q$, we see that there are 3 cases to consider:

1. $p^2 - 4q > 0$,
2. $p^2 - 4q < 0$,
3. $p^2 - 4q = 0$.

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Distinct real root

If $p^2 - 4q > 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct real roots λ_1 and λ_2.
The 2 functions $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are then solutions to the differential equation $y'' + py' + qy = 0$.
The Wronskian of y_1 and y_2 is then

$$\begin{aligned} W(t) &= y_1(t)y_2'(t) - y_2(t)y_1'(t) = \lambda_2 e^{\lambda_1 t} e^{\lambda_2 t} - \lambda_1 e^{\lambda_2 t} e^{\lambda_1 t} \\ &= (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)t} \neq 0 \end{aligned}$$

so y_1 and y_2 forms a fundamental set of solutions, and every solution of $y'' + py' + qy = 0$ has the form $c_1 y_1 + c_2 y_2$ where c_1 and c_2 are constants.

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Example

Let us find the general solution to the second-order differential equation $y'' - 3y' + 2y = 0$.

The characteristic polynomial is $\lambda^2 - 3\lambda + 2$ and the characteristic roots are $\lambda = \frac{3 \pm \sqrt{9-4 \cdot 2}}{2} = \frac{3 \pm 1}{2} = \begin{cases} 2 \\ 1 \end{cases}$.

It follows that the general solution is $y(t) = C_1 e^{2t} + C_2 e^{t}$.

Complex roots

If $p^2 - 4q < 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ has two distinct complex roots λ_1 and λ_2 which have the form $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ where a and b are real numbers and $b \neq 0$.
The 2 functions $z_1(t) = e^{\lambda_1 t} = e^{at}(\cos(bt) + i\sin(bt))$ and $z_2(t) = e^{\lambda_2 t} = e^{at}(\cos(bt) - i\sin(bt))$ are then solutions to the differential equation $y'' + py' + qy = 0$.
The two functions $y_1(t) = \mathrm{Re}(z_1(t)) = e^{at}\cos(bt)$ and $y_2(t) = \mathrm{Im}(z_1(t)) = e^{at}\sin(bt)$ are also solutions to the differential equation $y'' + py' + qy = 0$.

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Complex roots

The Wronskian of $y_1(t) = e^{at}\cos(bt)$ and $y_2 = e^{at}\sin(bt)$ is

$$\begin{aligned}
W(t) =& y_1(t)y_2'(t) - y_2(t)y_1'(t) \\
=& e^{at}\cos(bt)(ae^{at}\sin(bt) + be^{at}\cos(bt)) \\
& - e^{at}\sin(bt)(ae^{at}\cos(bt) - be^{at}\sin(bt)) \\
=& be^{2at}\cos^2(bt) + be^{2at}\sin^2(bt) \\
=& be^{2at} \neq 0
\end{aligned}$$

so y_1 and y_2 forms a fundamental set of solutions, and every solution of $y'' + py' + qy = 0$ has the form $c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$ where c_1 and c_2 are constants.

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Example

Let us find the general solution to the second-order differential equation $y'' + 2y' + 2y = 0$.

The characteristic polynomial is $\lambda^2 + 2\lambda + 2$, and the characteristic roots are $\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$.

It follows that the general solution is

$$y(t) = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t$$

Repeated roots

If $p^2 - 4q = 0$, then the characteristic polynomial $\lambda^2 + p\lambda + q$ just have one root $\lambda = -p/2$.
The function $y_1(t) = e^{\lambda t}$ is a solution to the differential equation $y'' + py' + qy = 0$.
Let $y_2(t) = te^{\lambda t}$. Then $y_2'(t) = e^{\lambda t} + \lambda te^{\lambda t}$,
$y_2''(t) = \lambda e^{\lambda t} + \lambda e^{\lambda t} + \lambda^2 te^{\lambda t}$ and

$$\begin{aligned} y_2''(t) + py_2'(t) + qy_2(t) &= 2\lambda e^{\lambda t} + \lambda^2 te^{\lambda t} + p(e^{\lambda t} + \lambda te^{\lambda t}) + qte^{\lambda t} \\ &= (2\lambda + p)e^{\lambda t} + (\lambda^2 + p\lambda + q)te^{\lambda t} \\ &= 0. \end{aligned}$$

So y_2 is a solution to the differential equation $y'' + py' + qy = 0$.

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Repeated roots

The Wronskian of $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ is

$$\begin{aligned} W(t) =& y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ =& e^{\lambda t}(e^{\lambda t} + \lambda te^{\lambda t}) - \lambda e^{\lambda t} te^{\lambda t} \\ =& e^{2\lambda t} \neq 0 \end{aligned}$$

so y_1 and y_2 forms a fundamental set of solutions, and every solution of $y'' + py' + qy = 0$ has the form $c_1 e^{\lambda t} + c_2 te^{\lambda t}$ where c_1 and c_2 are constants.

Example

Let us find the general solution to the second-order differential equation $y'' - 2y' + y = 0$.

The characteristic polynomial is $\lambda^2 - 2\lambda + 1 = 0$, and the characteristic roots are

$$\lambda = \frac{2 \pm \sqrt{2^2 - 4 \cdot 1}}{2} = 1.$$

It follows that the general solution is $y(t) = c_1 e^t + c_2 t e^t$.

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Exercise 4.3.26

Find the solution to the initial value problem

$$10y'' - y' - 3y = 0, \; y(0) = 1, \; y'(0) = 0.$$

The characteristic polynomial is $10\lambda^2 - \lambda - 3 = 0$ and the characteristic roots are

$$\lambda = \frac{1 \pm \sqrt{1 - 4 \cdot 10(-3)}}{2 \cdot 10} = \frac{1 \pm \sqrt{121}}{20}$$

$$= \frac{1 \pm 11}{20} = \begin{cases} 12/20 \\ -10/20 \end{cases} = \begin{cases} 3/5 \\ -1/2 \end{cases}$$

So the general solution to the differential equation is $y(t) = C_1 e^{3/5 t} + C_2 e^{-1/2 t}$.

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Exercise 4.3.26

Find the solution to the initial value problem

$$10y'' - y' - 3y = 0, \; y(0) = 1, \; y'(0) = 0.$$

Let $y(t) = c_1 e^{3/5 t} + c_2 e^{-1/2 t}$. Then $y'(t) = 3/5\, c_1 e^{3/5 t} - 1/2\, c_2 e^{-1/2 t}$, $y'(0) = 3/5\, c_1 - 1/2\, c_2$ and $y(0) = c_1 + c_2$. So y is a solution to the ivp if and only if $c_1 + c_2 = 1$ and $3/5\, c_1 - 1/2\, c_2 = 0$. $c_1 + c_2 = 1$ if and only if $c_2 = 1 - c_1$. If $c_2 = 1 - c_1$ then $3/5\, c_1 - 1/2\, c_2 = 0$ if and only if $3/5\, c_1 - 1/2 (1 - c_2)$

$$= \frac{3}{5} c_1 + \frac{1}{2} c_1 - 1/2 = 0$$ if and only if

$$1/2 = \left(\frac{3}{5} + \frac{1}{2}\right) c_1 = \frac{6+5}{10} c_1 = \frac{11}{10} c_1$$

we see that $c_1 = \frac{10}{22}$ and $c_2 = 12/22$

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Exercise 4.3.26

Find the solution to the initial value problem

$$10y'' - y' - 3y = 0, \; y(0) = 1, \; y'(0) = 0.$$

So the solution is $y(t) = \frac{10}{22} e^{3/5 t} + \frac{12}{22} e^{-1/2 t}$.