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EXAM IN TMA4110/4115 CALCULUS 3

English
Saturday, August 18, 2007
9 am – 1 pm

You may use the following (code C): Simple calculator (HP30S), with user's manual
Rottman: *Matematisk formelsamling*

Grades to be announced: September 10, 2007

All answers have to be justified. When grading, the 12 problems (1, 2abc, 3ab, 4ab, 5ab, 6, 7) will, as a rule, have the same weight.

Problem 1 Use polar form $z = re^{i\theta}$ to find the solutions of the equation

$$z^3 - 5\bar{z} = 0.$$

Problem 2

a) Solve the initial value problem

$$y'' - 2y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 4.$$

b) Find a general solution of the differential equation

$$y'' + y' - 12y = 7e^{-4x}.$$

c) Find a particular solution of the differential equation

$$y'' - \frac{2}{x^2}y = x^2, \qquad x > 0.$$

Problem 3

a) Solve the system

$$\begin{aligned} x_1 + x_2 + 3x_3 + 2x_4 &= 0 \\ 3x_1 - 2x_2 - x_3 - 4x_4 &= 0 \\ 4x_1 + x_2 + 6x_3 + 2x_4 &= 0. \end{aligned}$$

b) Let the 3×4 matrix A be given by

$$A = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 3 & -2 & -1 & -4 \\ 4 & 1 & 6 & 2 \end{bmatrix}.$$

Find a basis for each of the spaces $\text{Row}(A)$, $\text{Col}(A)$, $\text{Null}(A)$ and $\text{Col}(A)^\perp$.

Problem 4

a) A square 3×3 matrix A is given by

$$A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 1 & 0 & a \end{bmatrix}.$$

For which real numbers a is the matrix A invertible?

b) Find A^{-1} when $a = 1$.

Problem 5

a) Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}.$$

Write down an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

b) In a metropolitan area with a constant total population, 7 million people are now living in the city and 5 millions are living in the suburbs. Each year 20 % of the people in the city move to the suburbs (and 80 % remain in the city), while 10 % of the people living in the suburbs move into the city (and 90 % remain in the suburbs).

Find the long-term distribution of the population between the city and its suburbs.

Problem 6 It is given that 1 and 3 are the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let $x_1 = x_1(t)$, $x_2 = x_2(t)$ and $x_3 = x_3(t)$ be differentiable functions of t. Solve the system of differential equations

$$\begin{aligned} x_1' &= 2x_1 - x_2 + x_3 \\ x_2' &= -x_1 + 2x_2 + x_3 \\ x_3' &= 3x_3 \end{aligned}$$

with initial conditions $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = 1$.

Problem 7 Let A be an $m \times n$ matrix and B an $n \times p$ matrix such that $AB = 0$. Explain why Col(B) is contained in Null(A), and show that

$$\text{rank}(A) + \text{rank}(B) \leq n.$$