OVERSIKTSFORELESNING 8

Variabelskifte i trippelintegraler (14.6)

Eksempel: Finn volumet av ellipsoiden E gitt ved

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq R^2 \qquad (a, b, c > 0)$$

$$\text{volum}(E) = \iiint_E dV$$

Bytt til $u = \frac{x}{a}, \quad v = \frac{y}{b}, \quad w = \frac{z}{c}$

$$x = au, \quad y = bv, \quad z = cw$$

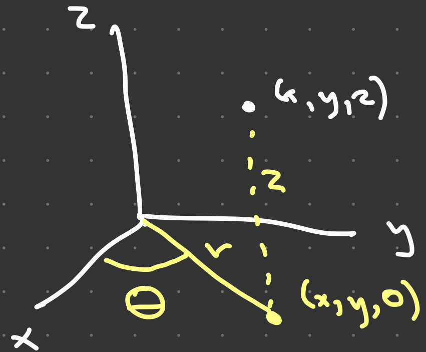
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc$$

$$\text{volum}(E) = \iiint_S abc \, du\,dv\,dw = abc \cdot \text{volum}(S)$$
$$= \frac{4}{3}\pi R^3 \cdot abc$$

$S = \{(u,v,w) : u^2 + v^2 + w^2 \leq R^2\}$ – sfære av radius R

Sylinderkoordinater (10.6 og 14.6)

Bruk polarkoordinater r, θ i xy-planet og behold z:



$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$

$$y = r\sin\theta \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$z = z \qquad z = z$$

For å ha én-én-tydighet bortefra z-aksen (for $(x,y) \neq (0,0)$) tar vi

$$r \geq 0, \quad 0 \leq \theta < 2\pi$$

<u>Eksempel:</u> Uttrykk $P_1 = (1,0,0)$, $P_2 = (0,1,h)$ i sylinderkoordinater

P_1: $r^2 = 1^2 + 0^2 \rightsquigarrow r = 1$

$1 = \cos\theta$

P_2: $r^2 = 0^2 + 1^2 \rightsquigarrow r = 1$

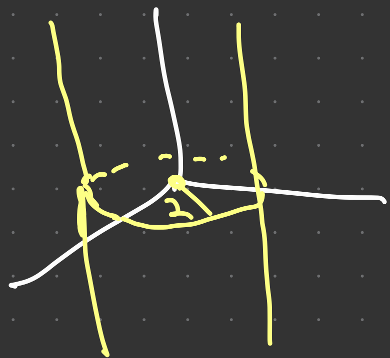
$0 = \cos\theta$

$0 = \sin\theta \rightsquigarrow \theta = 0$ $\qquad$ $1 = \sin\theta \rightsquigarrow \theta = \frac{\pi}{2}$

$z = 0$ $\qquad$ $z = h$

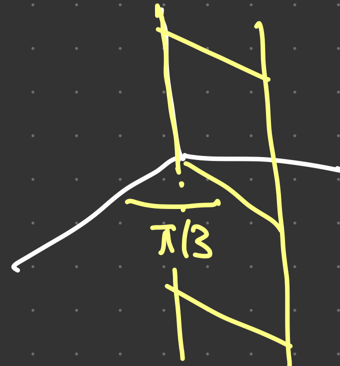
<u>Eksempel:</u> Flater i sylinderkoordinater:

<u>$r = 2$:</u>



- sylinder med radius 2 om z-aksen

$\theta = \frac{\pi}{3}$



$x = r\cos\theta = \frac{1}{2}r$

$y = r\sin\theta = \frac{\sqrt{3}}{2}r = \sqrt{3}\,x$

$r \geq 0$ gir: $x \geq 0$

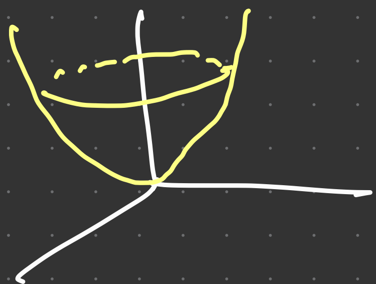
$y = \sqrt{3}\,x$

$z = 3$: Horisontalt plan med $z = 3$

$z = r - 1$:

$z = f(r)$: gitt ved rotasjon av grafen til f (for $r \geq 0$) om z-aksen

$z = r^2$



Volumelementet i sylinderkoordinater

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$\Rightarrow$ Hvis $T \subseteq \mathbb{R}^3$ tilsvarer S i (r, θ, z)-rommet har vi

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_S f(r\cos\theta, r\sin\theta, z)\,\underline{\underline{r}}\,dr\,d\theta\,dz$$

<u>Eksempel:</u> $T \subseteq \mathbb{R}^3$ er området over xy-planet og innenfor sylinderen $x^2+y^2=1$ og sfæren $x^2+y^2+z^2=4$

Finn $\iiint_T 2x^2 z\,dx\,dy\,dz$

T: $z \geq 0$

$x^2+y^2 \leq 1$

$x^2+y^2+z^2 \leq 4$

S: $z \geq 0$

$r \leq 1$

$r^2+z^2 \leq 4 \iff z \leq \sqrt{4-r^2}$

$$\iiint_T 2x^2 z \, dx\,dy\,dz = \iiint_S 2r^2 \cos^2\theta \, z \cdot r \, dz\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} 2r^3 \cos^2\theta \, z \, dz\,dr\,d\theta$$

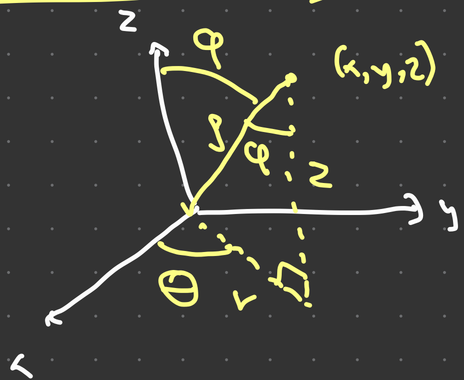
$$= \int_0^{2\pi} \int_0^1 r^3 \cos^2\theta \,(4-r^2)\, dr\,d\theta = \int_0^{2\pi} \cos^2\theta \left(r^4 - \frac{1}{6}r^6\right)_{r=0}^{1} d\theta$$

$$= \int_0^{2\pi} \frac{5}{6}\cos^2\theta \, d\theta = \frac{5}{12}\int_0^{2\pi} (\cos 2\theta + 1)\, d\theta = \frac{5}{12}\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\theta=0}^{2\pi}$$

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

$$= \frac{5\pi}{6}$$

Kulekoordinater (10.6 og 14.6)



ρ = avstand fra origo

φ = vinkel med z-aksen

θ = vinkel med x-aksen i xy-planet
(= samme som i sylinderkoord.)

$$r = \rho \sin\varphi$$
$$z = \rho \cos\varphi$$
$$x = r\cos\theta = \rho \sin\varphi \cos\theta$$
$$y = r\sin\theta = \rho \sin\varphi \sin\theta$$

Merk: Boken bruker R istedenfor ρ

For én-én-tydighet har vi

$\rho \geq 0$, $0 \leq \theta < 2\pi$

$0 \leq \varphi < \pi$

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$\varphi = \tan^{-1}\frac{r}{z} = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

Eksempel: Angi punktet $P = (0, -1, \sqrt{3})$ i kulekoordinater:

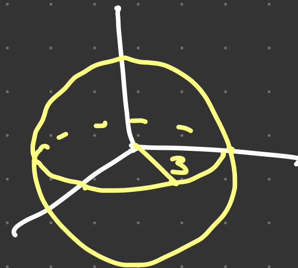
$$\rho = \sqrt{0+1+3} = 2$$

$$\theta = \frac{3\pi}{2}$$

$$\tan\varphi = \frac{r}{z} = \frac{1}{\sqrt{3}} \Longrightarrow \varphi = \frac{\pi}{6}$$

Eksempel: Flater i kulekoordinater

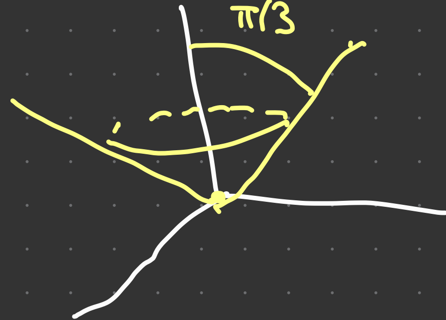
$\rho = 3$:



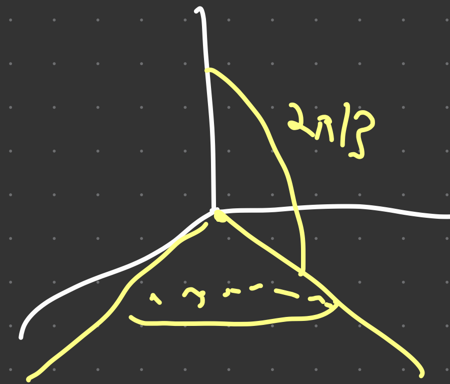
sfære med radius 3

$\Theta = \frac{\pi}{3}$: halvplan som for sylinderkoord.

$\varphi = \frac{\pi}{3}$



$\varphi = \frac{2\pi}{3}$



Volumelementet i kulekoordinater

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \det\begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\theta & 0 \end{pmatrix}$$

$$= \rho^2\sin\varphi \quad (\geq 0 \text{ siden } 0 \leq \varphi \leq \pi)$$

$\Rightarrow$ Hvis $T \subseteq \mathbb{R}^3$ tilsvarer S i (ρ, φ, θ)-rommet har vi

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_S f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

<u>Eksempel:</u> Finn volumet av sfæren $T: x^2+y^2+z^2 \leq R^2$

$$\text{volum}(T) = \iiint_T dV = \iiint_S \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$S = \{(\rho, \varphi, \theta) : \rho \leq R\}$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho^2 \sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{1}{3}R^3 \sin\varphi\,d\theta\,d\varphi$$

$$= \frac{2\pi}{3} R^3 \int_0^{\pi} \sin\varphi \, d\varphi = \frac{2\pi}{3} R^3 \left[-\cos\varphi \right]_{\varphi=0}^{\pi}$$

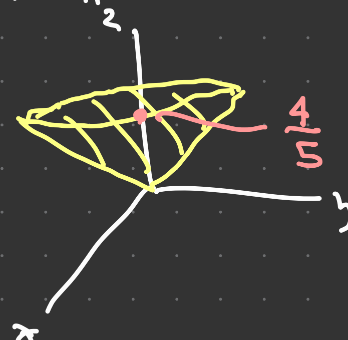
$$= \frac{2\pi}{3} R^3 (1+1) = \frac{4\pi}{3} R^3$$

Masse og massesenter (14.7)

Eksempel: Finn massesenteret til kjeglen begrenset av

$z = \sqrt{x^2+y^2} = r$

$0 \leq z \leq 1$



med δ proporsjonal med avstand fra origo,

$$\delta(x,y,z) = c\sqrt{x^2+y^2+z^2} = c\rho \quad \text{for } c>0$$

$$m = \iiint_T \delta(x,y,z)\,dx\,dy\,dz = \iiint_S c\rho\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$T = \left\{(x,y,z) : 0 \le z \le 1,\ 0 \le \sqrt{x^2+y^2} \le z\right\}$$

I kulekoordinater:

S gitt ved

$$0 \le \rho\sin\varphi \le \rho\cos\varphi \Leftrightarrow 0 \le \tan\varphi \le 1 \Longleftrightarrow 0 \le \varphi \le \frac{\pi}{4}$$

$$0 \le \rho\cos\varphi \le 1 \Longleftrightarrow 0 \le \rho \le \frac{1}{\cos\varphi}$$

$$m = \int_0^{\pi/4} \int_0^{1/\cos\varphi} \int_0^{2\pi} c\,\rho^3 \sin\varphi \, d\theta \, d\rho \, d\varphi$$

$$= 2\pi c \int_0^{\pi/4} \int_0^{1/\cos\varphi} \rho^3 \sin\varphi \, d\rho \, d\varphi$$

$$= 2\pi c \int_0^{\pi/4} \frac{1}{4} \frac{\sin\varphi}{\cos^4\varphi} \, d\varphi = \frac{\pi c}{2} \int_0^{\pi/4} \frac{\sin\varphi}{\cos^4\varphi} \, d\varphi$$

$$= \frac{\pi c}{2} \int_1^{1/\sqrt{2}} -\frac{1}{u^4} \, du = \frac{\pi c}{2} \int_{1/\sqrt{2}}^{1} \frac{1}{u^4} \, du = \frac{\pi c}{2} \left[-\frac{1}{3u^3} \right]_{u=1/\sqrt{2}}^{1}$$

$$= \frac{\pi c}{6} (2\sqrt{2} - 1)$$

$u = \cos\varphi$

$du = -\sin\varphi \, d\varphi$

$\varphi = 0 \longleftrightarrow u = 1$

$\varphi = \frac{\pi}{4} \longleftrightarrow u = 1/\sqrt{2}$

$$\bar{x} = \frac{1}{m}\iiint_T x\,\delta(x,y,z)\,dx\,dy\,dz$$

Rotasjonssymmetri om z-aksen $\Rightarrow$ massesenteret må ligge på z-aksen, $\bar{x}=0$, $\bar{y}=0$

$$\bar{x} = \frac{1}{m}\int_0^{\pi/4}\int_0^{1/\cos\varphi}\int_0^{2\pi} \rho\sin\varphi\cos\theta \cdot c\rho \cdot \rho^2\sin\varphi \, d\theta\, d\rho\, d\varphi$$

$$= \frac{1}{m}\int_0^{\pi/4}\int_0^{1/\cos\varphi} \underbrace{\Big[\sin\theta\Big]_{\theta=0}^{2\pi}}_{=0} c\rho^4\sin^2\varphi\, d\rho\, d\varphi$$

$$=0$$

$$\bar{y} = 0$$

$$\bar{z} = \frac{1}{m}\int_0^{\pi/4}\int_0^{1/\cos\varphi}\int_0^{2\pi} \underbrace{\rho\cos\varphi}_{z} \cdot c\rho \cdot \rho^2 \sin\varphi \, d\theta \, d\rho \, d\varphi$$

$$= \frac{1}{m}\cdot 2\pi c \int_0^{\pi/4}\int_0^{1/\cos\varphi} \rho^4 \cos\varphi \sin\varphi \, d\rho \, d\varphi$$

$$= \frac{2\pi c}{m}\int_0^{\pi/4} \frac{1}{5}\frac{\sin\varphi}{\cos^4\varphi} \, d\varphi = \frac{2\pi c}{5m}\cdot\frac{1}{3}\left(2\sqrt{2}-1\right)$$

$$= \frac{2\pi c}{15m}\left(2\sqrt{2}-1\right) = \frac{4}{5}$$

$$(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{4}{5}\right)$$