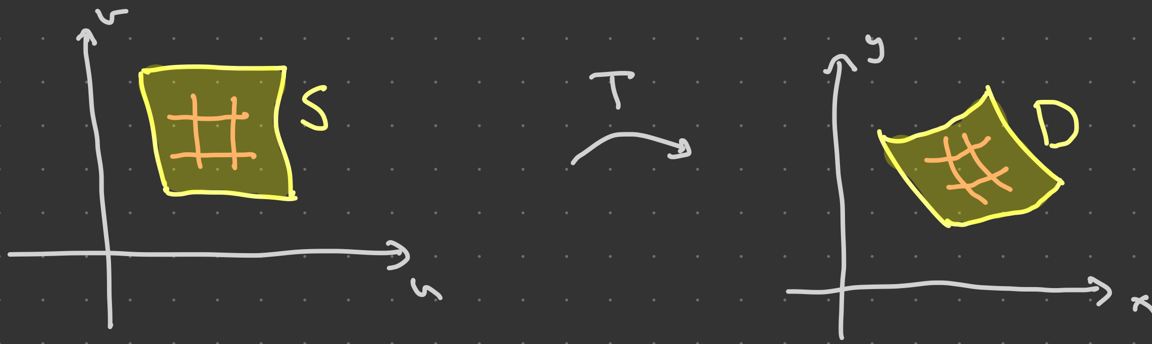
OVERSIKTSFORELESNING 7

Innlevering 2: 26.02

Variabelskifte i dobbeltintegraler (14.4)

I én variabel: $u = g(x)$

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$



Koordinattransformasjon: $x = x(u,v)$

$y = y(u,v)$

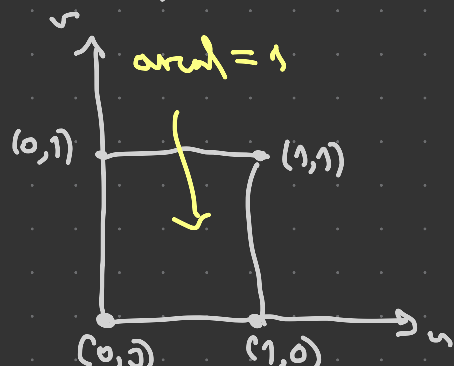
$T(u,v) = (x(u,v), y(u,v))$ fra S i uv-planet til D i xy-planet.

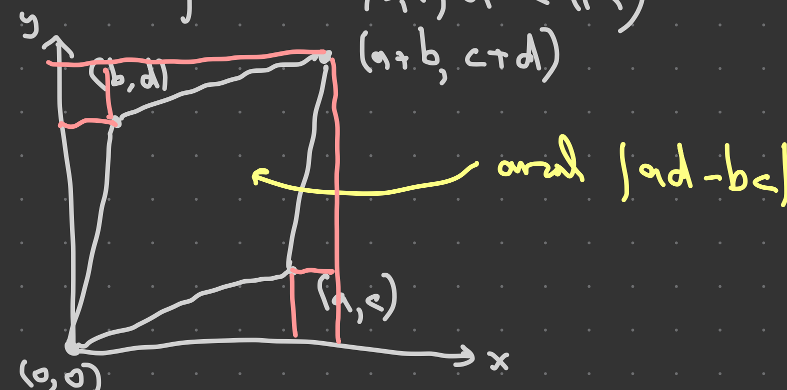
T er én-én-tydig fra S til D: for (x_0, y_0) i D finnes det et <u>unikt</u> punkt (u_0, v_0) i S slik at

$$T(u_0, v_0) = (x_0, y_0).$$

<u>Forklaring:</u>

$$T(u,v) = (au + bv,\ cu + dv) \quad (a,b,c,d \in \mathbb{R})$$





$$dA_{(x,y)} = |ad - bc|\, dA_{(u,v)}$$

For generell T: nær (u_0, v_0) har vi

$$T(u_0 + \Delta u, v_0 + \Delta v) \approx \left(x(u_0,v_0) + \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v, \; y(u_0,v_0) + \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v \right)$$

$$\Delta A_{(x,y)} \approx \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| \Delta A_{(u,v)}$$

Infinitesimalt nær punktet får vi

$$dA_{(x,y)} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{(u,v)}$$

<u>Merk</u>: I praksis har vi ofte $u(x,y)$, $v(x,y)$ eksplisitt og ønsker å bytte til u, v som variabler

$T: S \to D$ én-én-tydig $\Rightarrow$ T har en <u>invers</u>

$T(u,v) = (x(u,v), y(u,v))$

$T^{-1}: D \to S$

$T^{-1}(x,y) = (u(x,y), v(x,y))$

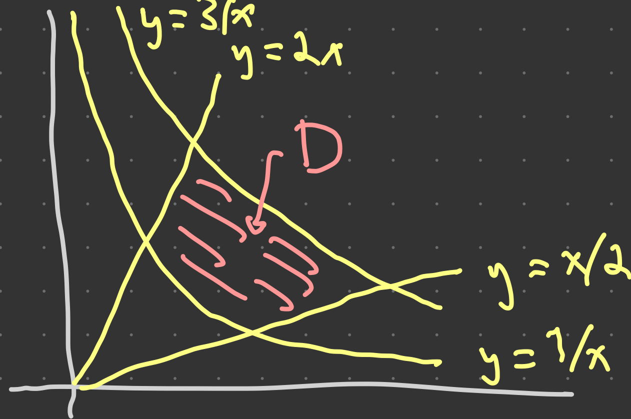
Hvis $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$ i D gir det implisitte funksjonsteoremet at T^{-1} har kontinuerlige partiell deriverte og

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$$

- kan finne Jacobi-determinanten ved å beregne $\frac{\partial(u,v)}{\partial(x,y)}$ uten å finne $x(u,v)$, $y(u,v)$ eksplisitt.

<u>Eksempel:</u> D = området begrenset av $y = \frac{1}{x}$, $y = \frac{3}{x}$, $y = \frac{x}{2}$, $y = 2x$.

$$f(x,y) = \frac{x}{y}$$



$$D = \left\{(x,y) : \frac{1}{x} \leq y \leq \frac{3}{x},\ \frac{1}{2}x \leq y \leq 2x\right\}$$

$$= \left\{(x,y) : 1 \leq xy \leq 3,\ \frac{1}{2} \leq \frac{y}{x} \leq 2\right\}$$

Hvis vi skifter til $u = xy$, $v = \frac{y}{x}$ tilsvarer D

$$S = \left\{(u,v) : 1 \leq u \leq 3,\ \frac{1}{2} \leq v \leq 2\right\}$$

$$\iint_D \frac{x}{y} \underbrace{dx\,dy}_{=\,dA_{(x,y)}} = \iint_S \underbrace{\frac{1}{v}}_{=\,\frac{x}{y}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \underbrace{du\,dv}_{=\,dA_{(u,v)}}$$

Enten: $x = \sqrt{\frac{u}{v}}, \quad y = \sqrt{uv} \qquad \left(\frac{u}{v} = xy \cdot \frac{x}{y} = x^2, \; uv = y^2\right)$

$$\frac{\partial x}{\partial u} = \frac{1}{2}\frac{1}{\sqrt{uv}}, \quad \ldots$$

Eller: $\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{x}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = y \cdot \frac{1}{x} - \left(-\frac{y}{x^2}\right) \cdot x = 2\frac{y}{x} = 2v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{2v}$$

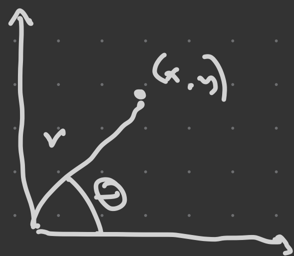
$v > 0$ i S, så v får

$$\iint_S \frac{1}{2v^2}\,du\,dv = \int_{\frac{1}{2}}^{2}\int_1^3 \frac{1}{2v^2}\,du\,dv$$

$$= \int_{\frac{1}{2}}^{2} (3-1)\cdot \frac{1}{2v^2}\,dv = \int_{\frac{1}{2}}^{2} \frac{1}{v^2}\,dv$$

$$= \left[-\frac{1}{v}\right]_{v=\frac{1}{2}}^{v=2} = 2 - \frac{1}{2} = \frac{3}{2}$$

Dobbeltintegraler i polarkoordinater (14.4)



$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r\cos^2\theta + r\sin^2\theta$$

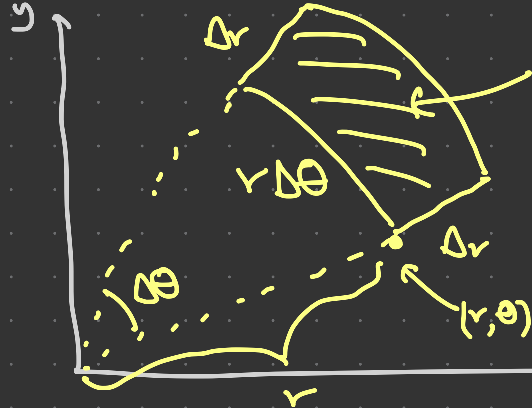
$$= r \quad (\geq 0)$$

Hvis D i xy-planet tilsvarer S i $r\theta$-planet har vi

$$\iint_D f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

<u>Merk</u>:

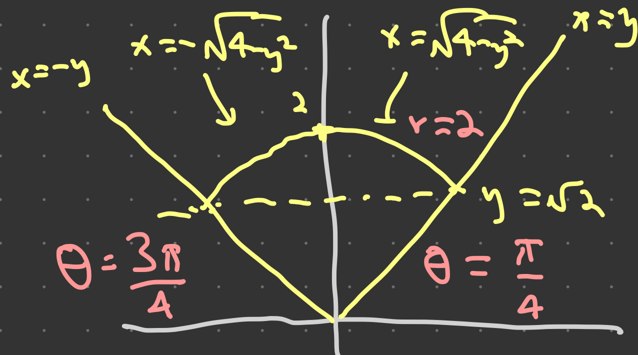
- For én-én-tydighet må vi begrense θ til (f.eks.) $0 \le \theta < 2\pi$
- $r=0$: $(r=0, \theta)$ beskriver $(0,0)$ i xy-planet for alle θ — dette er greit siden dobbelt integral over punkt/linje gir 0 (fordi areal $= 0$)



$$\Delta A \approx r\Delta\theta \cdot \Delta r$$

$$dx\,dy = dA_{(x,y)} = r\,d\theta\,dr$$

<u>Eksempel:</u> $$I = \int_0^{\sqrt{2}} \int_{-y}^{y} \ln(1+x^2+y^2)\,dx\,dy + \int_{\sqrt{2}}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \ln(1+x^2+y^2)\,dx\,dy$$



$$S = \left\{ (r,\theta) : 0 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4} \right\}$$

$$I = \iint_S \ln(1+r^2)\,r\,dr\,d\theta$$

$$I = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \ln(1+r^2)\, r \, d\theta \, dr$$

$$= \frac{\pi}{2} \int_0^2 \ln(1+r^2)\, r \, dr = \frac{\pi}{2} \int_1^5 \frac{1}{2} \ln u \, du = \frac{\pi}{4} \Big[u \ln u - u \Big]_{u=1}^{u=5}$$

$$= \frac{\pi}{4} \left(5 \ln 5 - 4 \right)$$

$u = 1 + r^2$

$du = 2r\, dr$

$r = 0 \Leftrightarrow u = 1$

$r = 2 \Leftrightarrow u = 5$

Eksempel: $I = \int_{-\infty}^{\infty} e^{-x^2} \, dx$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx\, dy$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\, r \, dr \, d\theta = \int_0^{2\pi} \int_0^{\infty} \frac{1}{2} e^{-u} \, du \, d\theta = \pi$$

$u = r^2$

$du = 2r\,dr$

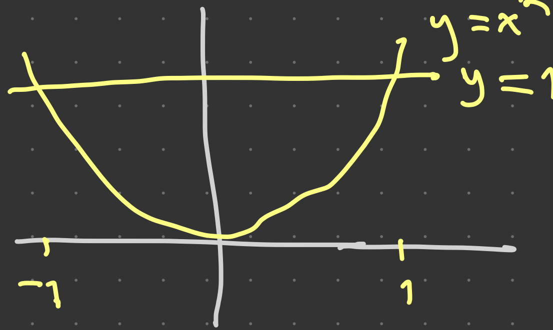
$$I = \sqrt{\pi}$$

Trippelintegraler (14.5)

<u>Eksempel</u>: Finn volumet avgrenset av $y = x^2$, $z = 0$,

$y + z = 1$

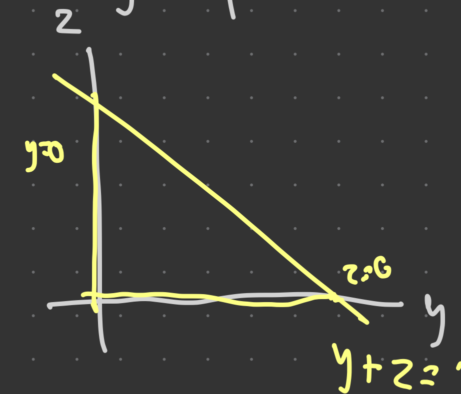
1 xy-planet $(z=0)$



$-1 \le x \le 1$

$x^2 \le y \le 1$

1 yz-planet $(x=0)$



$0 \le z \le 1-y$

$$D = \{(x,y,z) : -1 \le x \le 1,\ x^2 \le y \le 1,\ 0 \le z \le 1-y\}$$

$$\text{volum}(D) = \iiint_D 1\, dV = \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} 1 \, dz\, dy\, dx$$

$$= \int_{-1}^{1}\int_{x^2}^{1} (1-y)\,dy\,dx = \int_{-1}^{1} \left[y - \frac{1}{2}y^2 \right]_{y=x^2}^{1} dx$$

$$= \int_{-1}^{1} \left(\frac{1}{2} - x^2 + \frac{1}{2}x^4 \right) dx = \left[\frac{1}{2}x - \frac{1}{3}x^3 + \frac{1}{10}x^5 \right]_{-1}^{1}$$

$$= \frac{8}{15}$$

Eksempel: Regn ut $I = \int_0^1 \int_z^1 \int_0^x e^{x^3}\,dy\,dx\,dz$

ved å bytte integrasjonsrekkefølgen.

$$D = \{(x,y,z) : 0 \le z \le 1,\ z \le x \le 1,\ 0 \le y \le x\}$$

$$= \{(x,y,z) : 0 \le x \le 1,\ 0 \le z \le x,\ 0 \le y \le x\}$$

$$I = \int_0^1 \int_0^x \int_0^x e^{x^3}\, dy\, dz\, dx = \int_0^1 \int_0^x x\, e^{x^3}\, dz\, dx$$

$$= \int_0^1 x^2 e^{x^3}\, dx$$

$$u = x^3$$

$$du = 3x^2\, dx$$

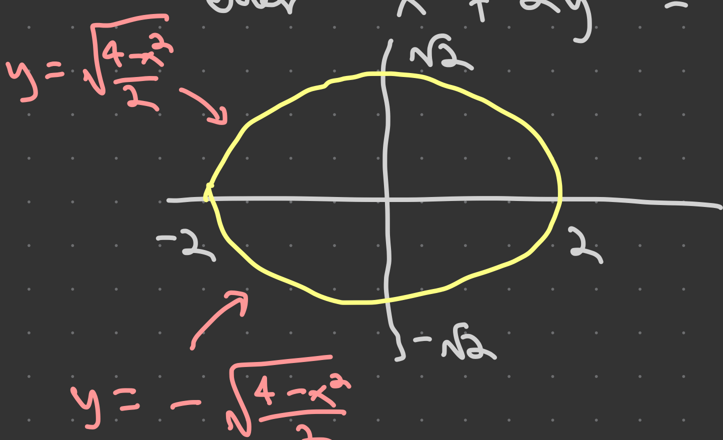
$$= \frac{1}{3} \int_0^1 e^u\, du = \frac{1}{3}(e-1)$$

Eksempel: Finn volumet av området D i $\mathbb{R}^3$ avgrenset av $z = x^2 + 3y^2$, $z = 8 - x^2 - y^2$

Projeksjonen av skjæringskurven i xy-planet er

$$x^2 + 3y^2 = 8 - x^2 - y^2$$

eller $$x^2 + 2y^2 = 4$$



$$-2 \le x \le 2$$

$$-\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}}$$

For $x^2+2y^2 \le 4$ har vi $8-x^2-y^2 \ge x^2+3y^2$

– flaten $z=8-x^2-y^2$ ligger over $z=x^2+3y^2$

$$D=\left\{(x,y,z):\ -2\le x\le 2,\ -\sqrt{\frac{4-x^2}{2}}\le y\le \sqrt{\frac{4-x^2}{2}},\ x^2+3y^2\le z\le 8-x^2-y^2\right\}$$

$$\text{volum}(D)=\iiint_D 1\,dV=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\left(8-2x^2-4y^2\right)dy\,dx$$

$$= \int_{-2}^{2} \left[(8-2x^2)y - \frac{4}{3}y^3 \right]_{y=-\sqrt{\frac{4-x^2}{2}}}^{y=\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \frac{4\sqrt{2}}{3}(4-x^2)^{3/2}\, dx = \cdots = 8\pi\sqrt{2}$$

Bytt til $x = 2\sin u$

$dx = 2\cos u \, du$