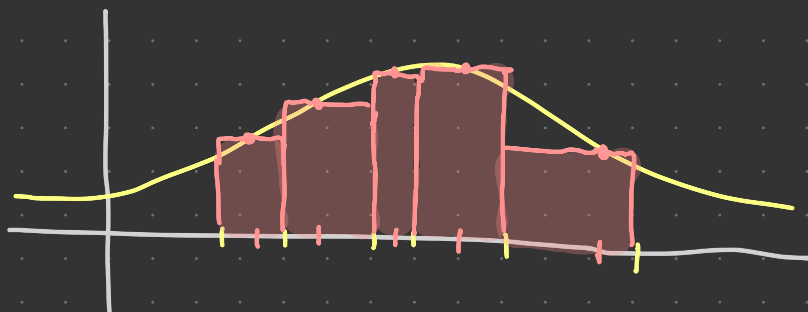
OVERSIKTSFORELESNING 6

Doble integraler (14.1)



I én variabel: $f : [a,b] \to \mathbb{R}$

En partisjon av $[a,b]$ er $a = x_0 < x_1 < \cdots < x_n = b$

$\Delta x_i = x_i - x_{i-1}$

Riemann-summen for f: $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$, x_i^* et vilkårlig punkt i $[x_{i-1}, x_i]$

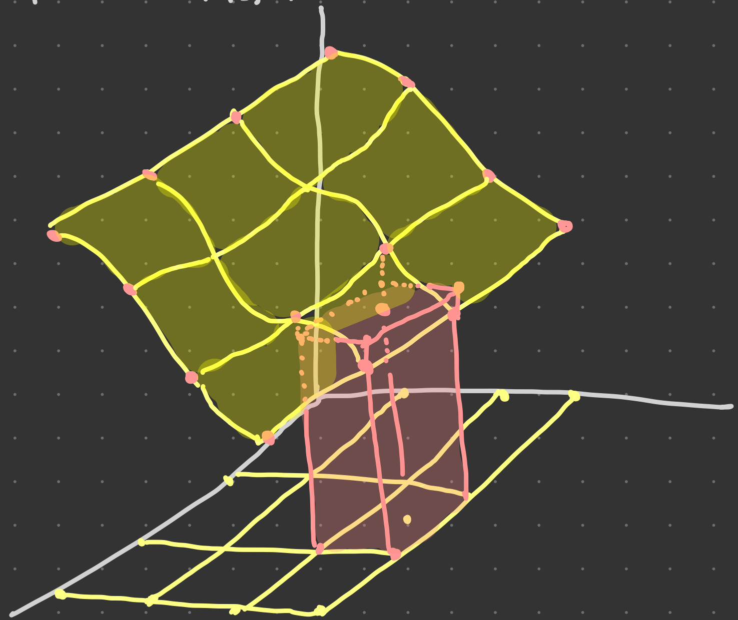
Normen av P: $\|P\| = \max_i \Delta x_i$

Integralet $\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_i f(x_i^*)\Delta x_i$

- hvis denne grensen eksisterer

- i så fall er f integrerbar på $[a,b]$

1 to variables:



$\iint_D$ for generell D

$D \subseteq \mathbb{R}^2$ lukket og begrenset

Vi kan velge et rektangel R slik at $D \subseteq R$

$f : D \longrightarrow \mathbb{R}$

Definér $\hat{f} : R \longrightarrow \mathbb{R}$ ved $\hat{f}(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & \text{ellers} \end{cases}$

Def: $\iint_D f(x,y)\,dA = \iint_R \hat{f}(x,y)\,dA$

hvis dette eksisterer

[Dette er uavhengig av valg av R)

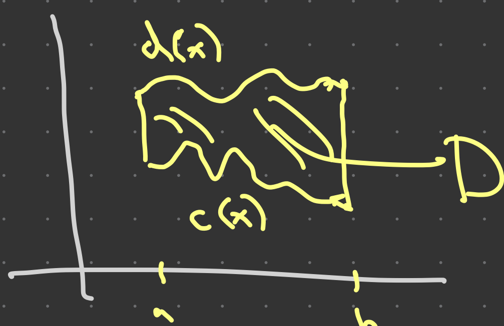
f integrerbar på D $\underbrace{\Longleftrightarrow}_{\text{hvis og bare hvis}}$ $\hat{f}$ integrerbar på R

Itererte integraler (14.2)

Def: $D \subseteq \mathbb{R}^2$ er *y-enkelt* dersom

$$D = \{(x,y) : a \leq x \leq b, \quad c(x) \leq y \leq d(x)\}$$

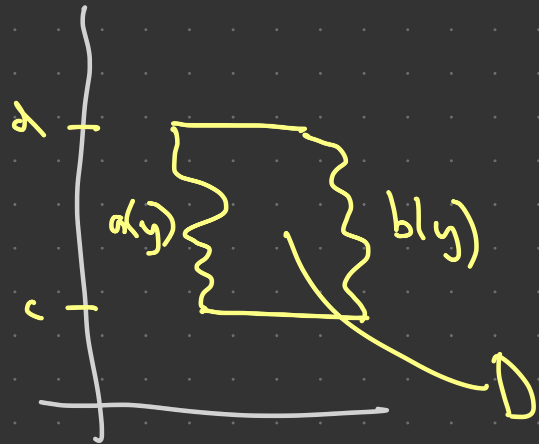
der $c, d : [a,b] \to \mathbb{R}$ er kontinuerlige kurver og $c(x) \leq d(x)$ for alle x

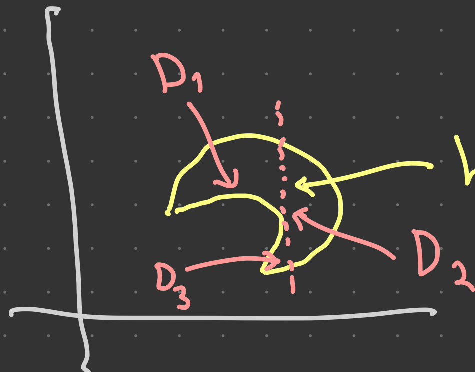


Def: $D \subseteq \mathbb{R}^2$ er <u>x-enkelt</u> hvis

$$D = \{(x,y) : c \le y \le d, \quad a(y) \le x \le b(y)\}$$

$a, b : [c,d] \to \mathbb{R}$ kontinuerlige, $a(y) \le b(y)$ for alle y

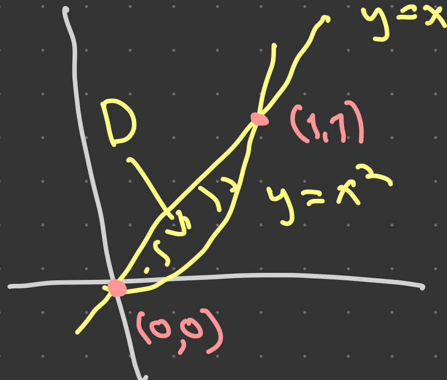




<u>Merk:</u> Hvis D er både x-enkelt og y-enkelt kan vi skrive $\iint_D f(x,y)\,dA$ på to måter som itererte integraler — dette lar oss bytte rekkefølgen i noen itererte integraler.

Eksempel:

D = området mellom $y=x^2$ og $y=x$



$$D = \{(x,y) : 0 \leq x \leq 1,\ x^2 \leq y \leq x\}$$

$$= \{(x,y) : 0 \leq y \leq 1,\ y \leq x \leq \sqrt{y}\}$$

– både x- og y-enkelt

$$f(x,y) = x^2 y$$

$$\iint_D f(x,y)\,dA = \int_0^1 \left(\int_{x^2}^{x} x^2 y \, dy \right) dx = \int_0^1 \left[\frac{1}{2} x^2 y^2 \right]_{y=x^2}^{x} dx$$

$$= \frac{1}{2} \int_0^1 (x^4 - x^6)\,dx = \left[\frac{1}{2} \left(\frac{1}{5} x^5 - \frac{1}{7} x^7 \right) \right]_0^1$$

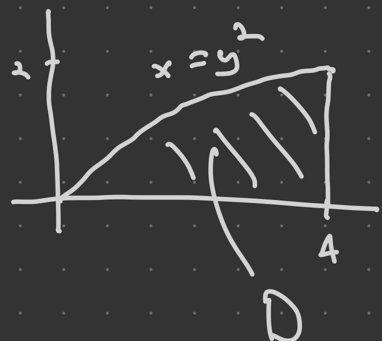
$$= \frac{1}{35}$$

$$\iint_D f(x,y)\,dA = \int_0^1 \int_y^{\sqrt{y}} x^2 y \, dx\, dy = \int_0^1 \left[\frac{1}{3} x^3 y \right]_{x=y}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \frac{1}{3} \left(y^{5/2} - y^4 \right) dy = \frac{1}{3} \left[\frac{2}{7} y^{7/2} - \frac{1}{5} y^5 \right]_0^1 = \frac{1}{35}$$

__Eksempel:__ Finn $I = \int_0^2 \int_{y^2}^4 \cos x^{3/2} \, dx \, dy$

ved å bytte integrasjonsrekkefølgen.



$D = \{(x,y) : 0 \le y \le 2, \ y^2 \le x \le 4\}$

$= \{(x,y) : 0 \le x \le 4, \ 0 \le y \le \sqrt{x}\}$

$$I = \int_0^4 \int_0^{\sqrt{x}} \cos x^{3/2} \, dy \, dx$$

$$= \int_0^4 \Big[y \cos x^{3/2} \Big]_{y=0}^{y=\sqrt{x}} dx = \int_0^4 \sqrt{x} \cos x^{3/2} \, dx$$

$u = x^{3/2}$

$du = \frac{3}{2} x^{1/2}\, dx$

$x = 4 \longleftrightarrow u = 8$

$x = 0 \longleftrightarrow u = 0$

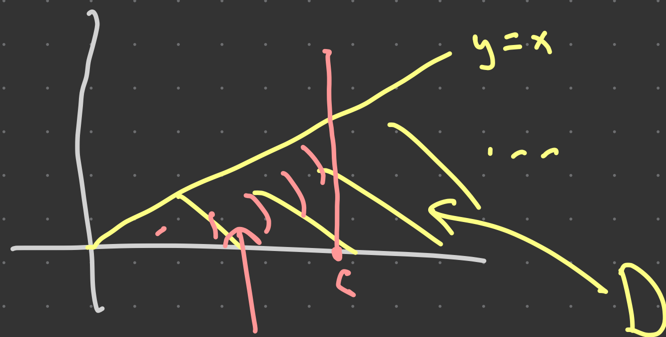
$$= \int_0^8 \frac{2}{3} \cos u \, du = \frac{2}{3} \sin 8$$

Uegentlige integraler (14.3)

Eksempel:

$D = \{ (x,y) : x \geq 0, \ 0 \leq y \leq x \}$

$f(x,y) = y e^{-x}$



$$D_c = \{(x,y) : 0 \le x \le c,\ 0 \le y \le x\}$$

$$\iint_{D_c} f(x,y)\, dA = \int_0^c \int_0^x y e^{-x}\, dy\, dx$$

$$= \int_0^c \left[\frac{1}{2} y^2 e^{-x}\right]_{y=0}^{x} dx = \int_0^c \frac{1}{2} x^2 e^{-x}\, dx$$

$$= \left[-\frac{1}{2}(x^2+2x+2)e^{-x} \right]_0^c$$

$$= \frac{1}{2}\left(2-(c^2+2c+2)e^{-c}\right)$$

Dersom $\lim\limits_{c\to\infty} \iint_{D_c} f(x,y)\,dA$ eksisterer, sier vi at

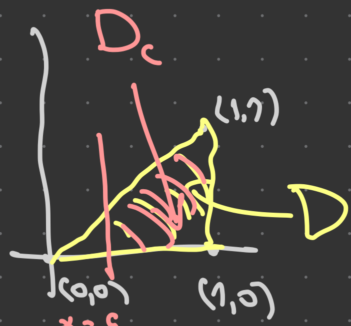
$\iint_D f(x,y)\,dA$ konvergerer, og er gitt ved grenseverdien

Her er $\lim\limits_{c\to\infty} \frac{1}{2}\left(2-(c^2+2c+2)e^{-c}\right) = 1$

$$\iint_D f(x,y)\, dA = 1$$

Eksempel: La D være trekanten med hjørner $(0,0)$, $(1,0)$, $(1,1)$

$$f(x,y) = \frac{1}{(x+y)^2}$$



$$D_c = \{(x,y) : c \le x \le 1,\ 0 \le y \le x\}$$

$$\iint_{D_c} f(x,y)\, dA = \int_c^1 \int_0^x \frac{1}{(x+y)^2}\, dy\, dx = \int_c^1 \left[-\frac{1}{x+y}\right]_{y=0}^{x} dx$$

$$= \int_c^1 \left[-\frac{1}{2x} + \frac{1}{x} \right] dx = \int_c^1 \frac{1}{2x} dx$$

$$= \left[\frac{1}{2} \ln x \right]_{x=c}^{1} = -\frac{1}{2} \ln c$$

$$\lim_{c \to 0^+} \iint_{D_c} f(x,y)\, dA = \lim_{c \to 0^+} -\frac{1}{2} \ln c = \infty$$

$$\Longrightarrow \iint_D f(x,y)\, dA \quad \underline{\text{divergerer}}$$

Middelverdier (14.3)

Def: $D \subseteq \mathbb{R}^2$ lukket og begrenset

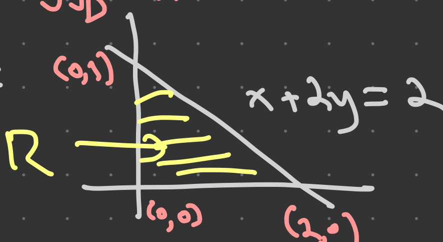
$f : D \to \mathbb{R}$ integrerbar

Middelverdien (gjennomsnittsverdien) til f på D er

$$\bar{f} = \frac{1}{\text{areal}(D)} \iint_D f(x,y)\, dA$$

$\text{areal}(D) = \iint_D 1\, dA$

Eksempel:



Tykkelsen til en metallplate over R varierer som

$f(x,y) = xy + 1$

Finn gjennomsnittlig tykkelse.

$$\text{areal}(R) = \frac{1}{2} \cdot 1 \cdot 2 = 1$$

$$\bar{f} = \frac{1}{\text{areal}\, R} \iint_R f(x,y)\, dA = \int_0^1 \int_0^{2-2y} (xy+1)\, dx\, dy$$

$$= \int_0^1 \left[\frac{1}{2} x^2 y \right]_{x=0}^{2-2y} dy + \underbrace{1}_{\iint_R 1\, dx\, dy = \text{areal}(R)}$$

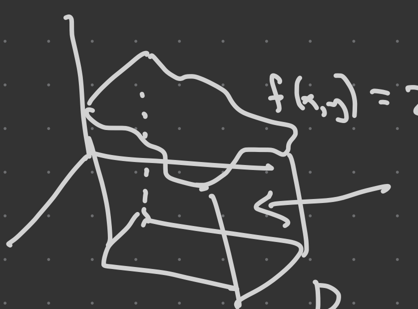
$$= \int_0^1 \frac{1}{2} (2-2y)^2 y\, dy \quad + 1$$

$$= \int_0^1 2(1-y)^2 y \, dy + 1$$

$$= \int_0^1 2(y - 2y^2 + y^3) \, dy + 1$$

$$= 2\left[\frac{1}{2}y^2 - \frac{2}{3}y^3 + \frac{1}{4}y^4\right]_0^1 + 1$$

$$= 2\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] + 1 = \frac{7}{6}$$



$\iint_D f(x,y)\, dA$ = volum