Matematikk 2
Oversiktsforelesninger uke 10

Frode Rønning

Nøkkelbegreper uke 10

- (Litt mer om Greens teorem og Divergensteoremet)
- Jacobi-determinanten
- Trippelintegraler og itererte integraler i tre dimensjoner
- Variabelskifte i trippelintegraler
  - til/fra sfæriske, sylindriske og kartesiske koordinater (volumelementet dV gitt ved kartesiske, sylindriske og sfæriske koordinater)
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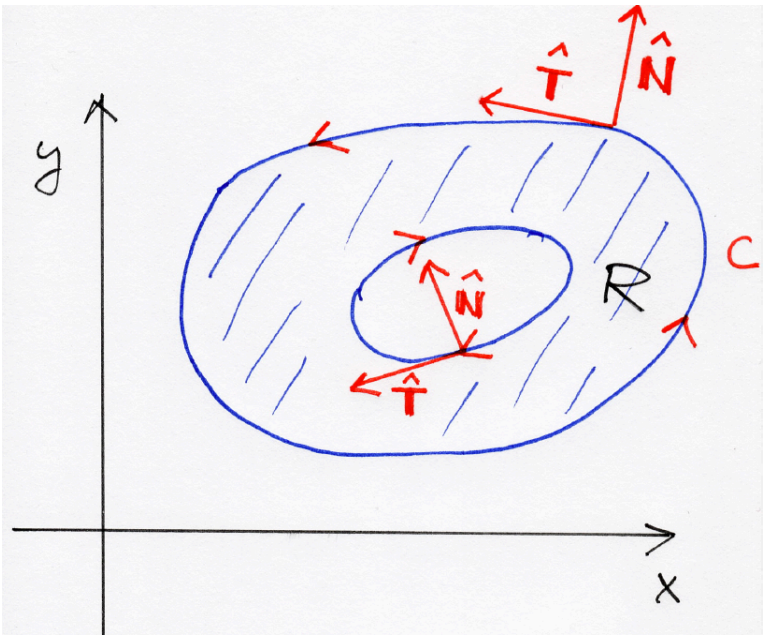
R et regulært, lukket område i xy-planet med rand C som består av stykkevis glatte, lukkede kurver, positivt orientert i forhold til R. La $\mathbf{F}(x,y) = F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}$ være et glatt vektorfelt på R

$\hat{\mathbf{T}} = T_1(x,y)\mathbf{i} + T_2(x,y)\mathbf{j}$: enhetstangentvektor til C i positiv omløpsretning

$\hat{\mathbf{N}} = T_2(x,y)\mathbf{i} - T_1(x,y)\mathbf{j}$: ytre enhetsnormalvektor til C

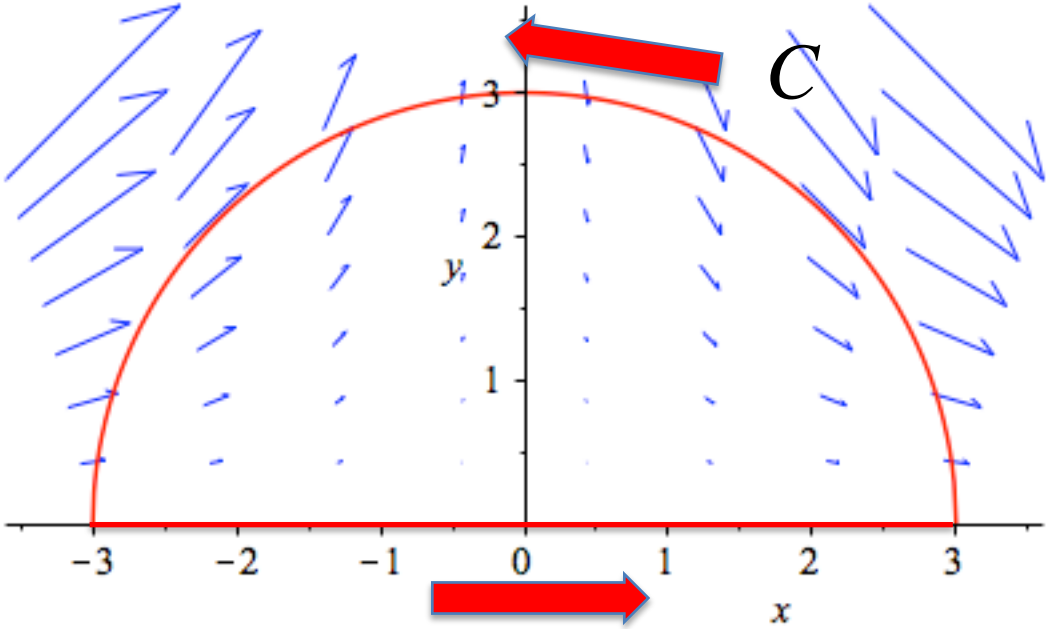


Greens teorem

$$\oint_C \mathbf{F} \bullet \hat{\mathbf{T}}\, ds = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

Divergensteoremet

$$\oint_C \mathbf{F} \bullet \hat{\mathbf{N}}\, ds = \iint_R \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dA$$

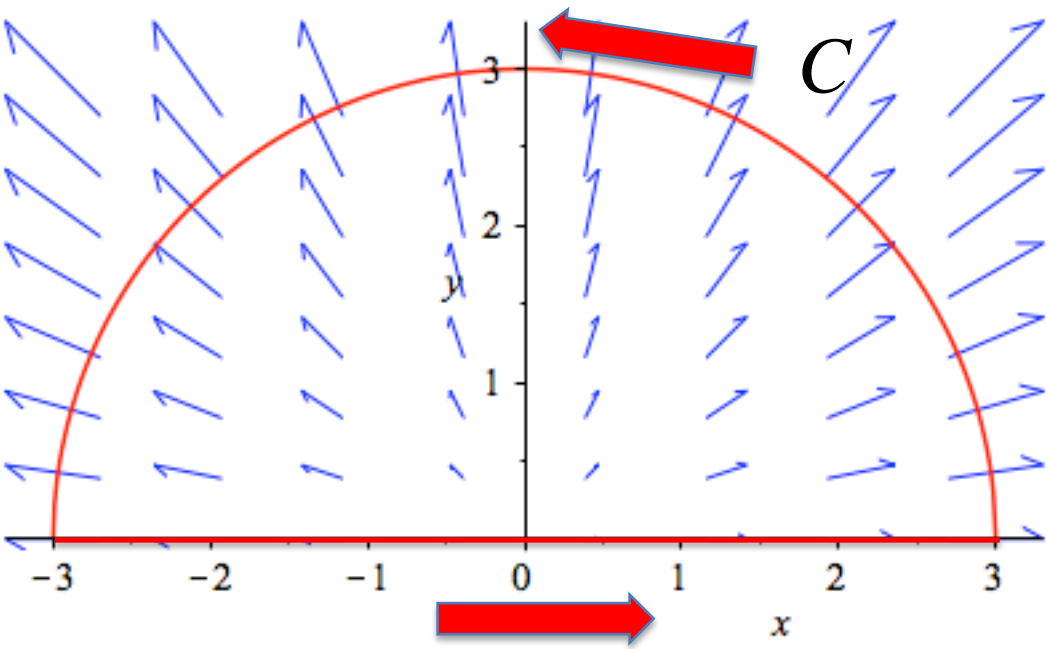


Eks. 1

$$\mathbf{F}(x,y) = x^2 y\mathbf{i} - xy^2\mathbf{j}$$

$$\oint_C \mathbf{F}\bullet\hat{\mathbf{T}}\,ds = -\frac{81}{4}\pi$$

$$\oint_C \mathbf{F}\bullet\hat{\mathbf{N}}\,ds = 0$$



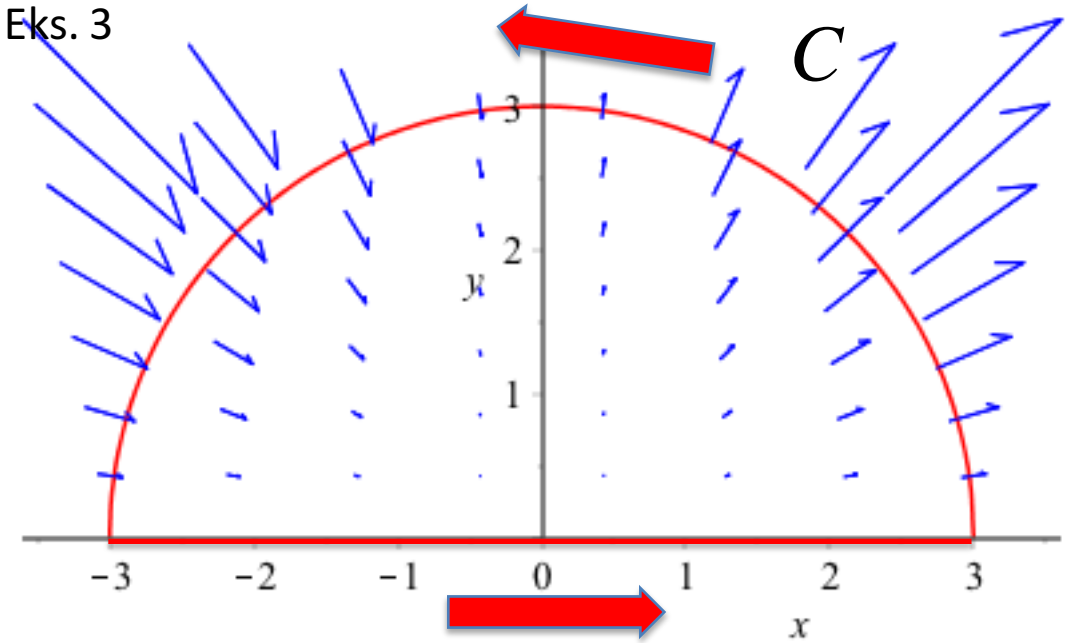
Eks. 2

$$\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$$

$$\oint_C \mathbf{F}\bullet\hat{\mathbf{T}}\,ds = 0$$

$$\oint_C \mathbf{F}\bullet\hat{\mathbf{N}}\,ds = 9\pi$$

Eks. 3



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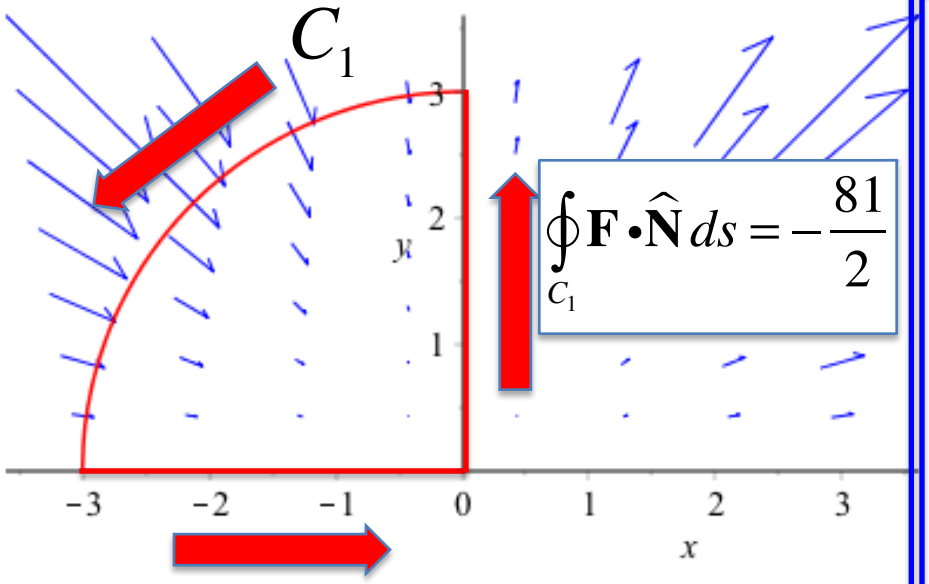
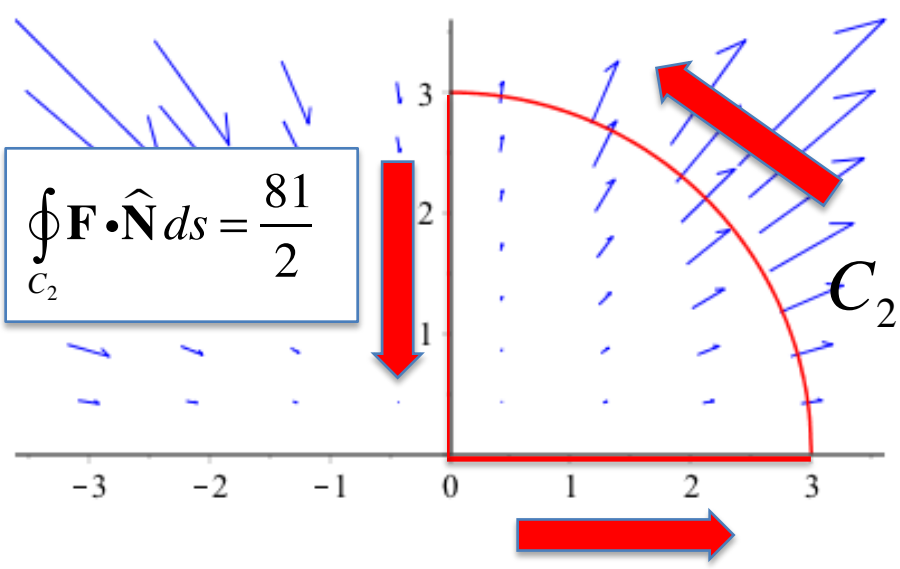


$$\oint_{C_1} \mathbf{F}\bullet\hat{\mathbf{N}}\,ds = -\frac{81}{2}$$

$$\oint_{C_2} \mathbf{F}\bullet\hat{\mathbf{N}}\,ds = \frac{81}{2}$$



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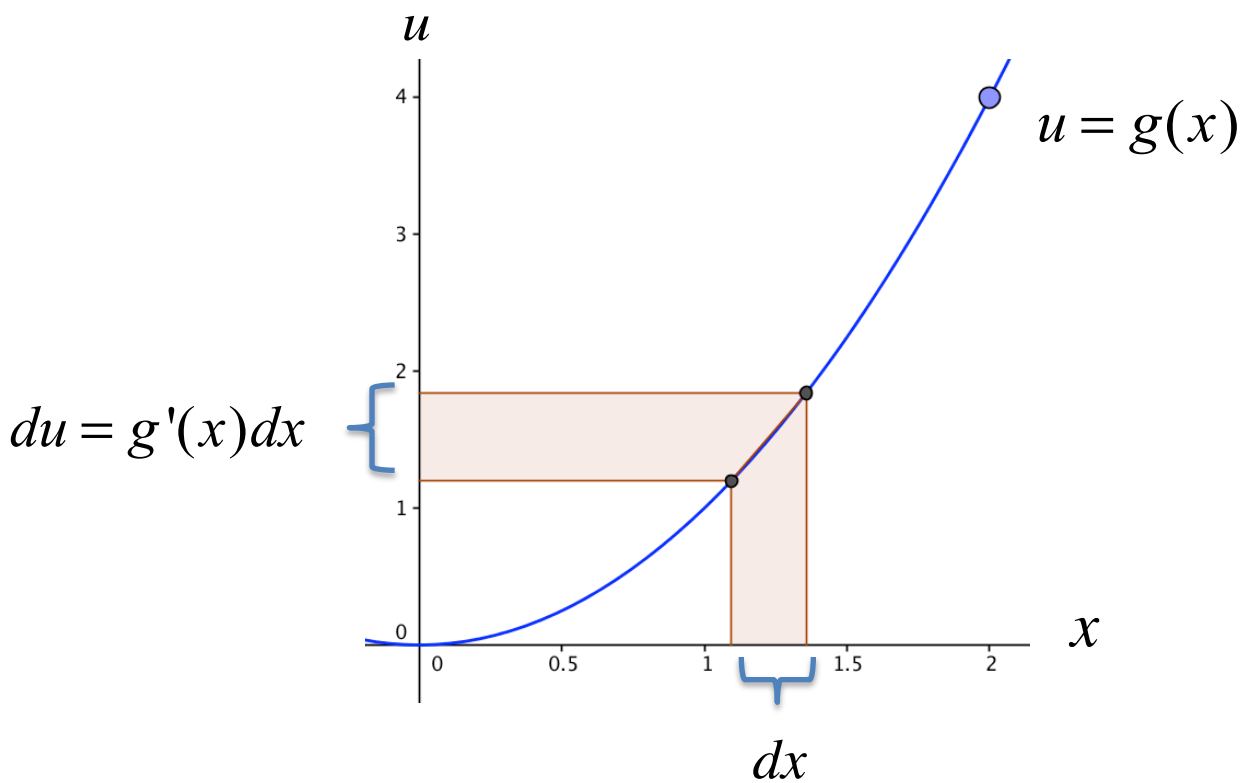
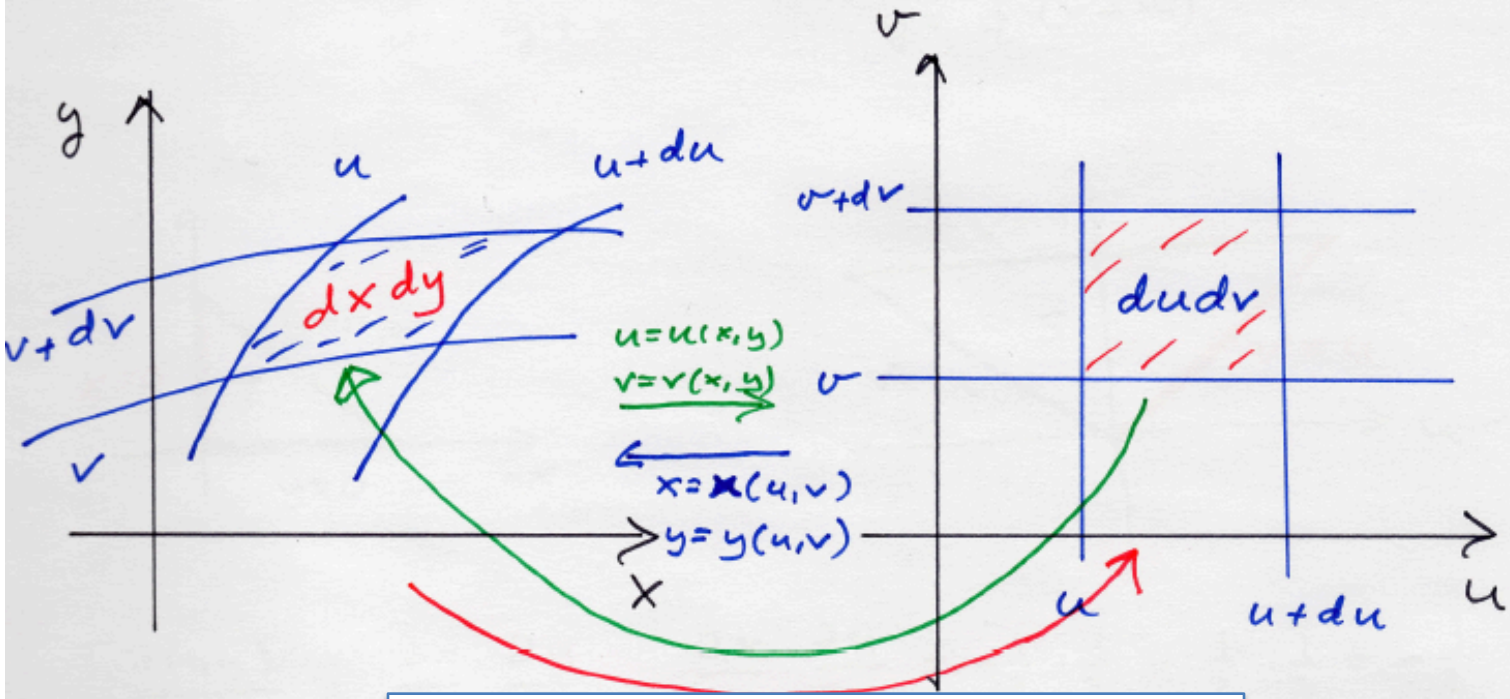
Variabelskifte i enkeltintegral

$$u = g(x)$$
$$du = g'(x)dx$$

$$x = g^{-1}(u)$$
$$dx = \frac{1}{g'(x)}du = (g^{-1})'(u)du$$



Variabelskifte i dobbeltintegral



$$dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$dudv = \frac{1}{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|} dxdy = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dxdy$$

Jacobi-determinanten. (Eksempel fra Interaktiv forelesning uke 9)

$$u = x^2 - y^2 \qquad (1)$$
$$v = xy$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 2x^2 + 2y^2 = 2(x^2 + y^2)$$

Implisitt derivasjon av (1) med hensyn på u gir: (se Notat om skalarfelt s. 20)

$$1 = 2x\frac{\partial x}{\partial u} - 2y\frac{\partial y}{\partial u}$$
$$0 = \frac{\partial x}{\partial u}y + x\frac{\partial y}{\partial u}$$

Løs dette likningssystemet:

$$\frac{\partial x}{\partial u} = \frac{x}{2(x^2 + y^2)}$$
$$\frac{\partial y}{\partial u} = \frac{-y}{2(x^2 + y^2)}$$

Implisitt derivasjon av (1) med hensyn på v gir:

$$0 = 2x\frac{\partial x}{\partial v} - 2y\frac{\partial y}{\partial v}$$
$$1 = \frac{\partial x}{\partial v}y + x\frac{\partial y}{\partial v}$$

Løs dette likningssystemet:

$$\frac{\partial x}{\partial v} = \frac{y}{x^2 + y^2}$$
$$\frac{\partial y}{\partial v} = \frac{x}{x^2 + y^2}$$

Jacobi-determinanten. (Eksempel fra Interaktiv forelesning uke 9)

$$u = x^2 - y^2 \qquad (1)$$
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$$\frac{\partial x}{\partial u} = \frac{x}{2(x^2+y^2)}$$
$$\frac{\partial y}{\partial u} = \frac{-y}{2(x^2+y^2)}$$

og

$$\frac{\partial x}{\partial v} = \frac{y}{x^2+y^2}$$
$$\frac{\partial y}{\partial v} = \frac{x}{x^2+y^2}$$

gir:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{x}{2(x^2+y^2)} \cdot \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} \cdot \frac{-y}{2(x^2+y^2)} = \frac{1}{2(x^2+y^2)}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$$

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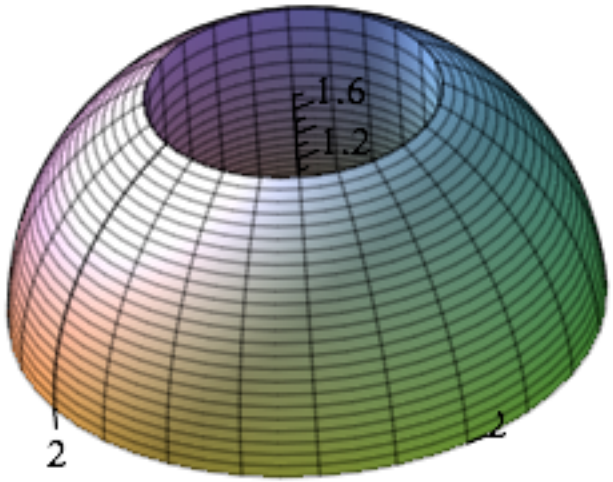
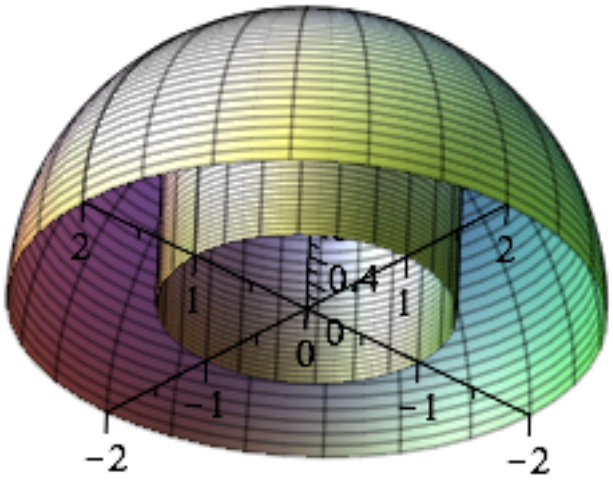
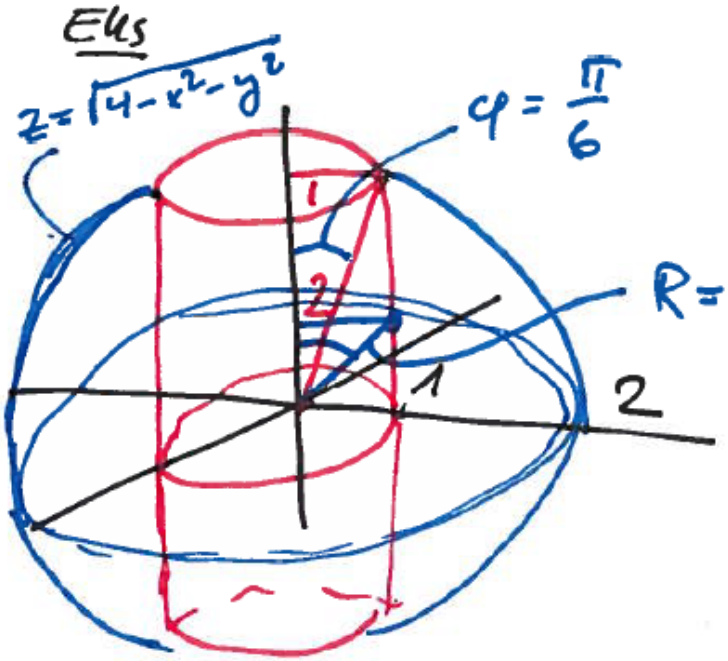




Halvkule med radius 2, skjært gjennom av sylinder med radius 1

Eks



Kule med radius 2

Sylinder med radius 1

$$V = 2\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_{\frac{1}{\sin\varphi}}^{2} R^2 \sin\varphi \, dR \, d\varphi \, d\theta$$

$$= \frac{4}{3}\pi \int_{\pi/6}^{\pi/2} R^3 \sin\varphi \Big|_{\frac{1}{\sin\varphi}}^{2} d\varphi$$

$$= \frac{4}{3}\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin\varphi - \frac{1}{\sin^2\varphi}\right) d\varphi$$

$$= \frac{32}{3}\pi \left[-\cos\varphi\right]_{\pi/6}^{\pi/2} + \frac{4}{3}\pi \left[\cot\varphi\right]_{\pi/6}^{\pi/2} = \frac{32}{3}\pi \cdot \frac{1}{2}\sqrt{3} - \frac{4}{3}\pi\sqrt{3} = 4\pi\sqrt{3}$$

Utregnet med kulekoordinater

Med sylinderkoordinater:

$$V = 2\int_0^{2\pi}\int_1^2\int_0^{\sqrt{4-r^2}} z\,dz\,r\,dr\,d\theta$$

$$= 4\pi\int_1^2 \sqrt{4-r^2}\,r\,dr = 4\pi\left(-\frac{1}{3}(4-r^2)^{3/2}\right)_{r=1}^{2} = \frac{4\pi}{3}\,3^{3/2}$$

$$= \underline{4\pi\sqrt{3}}$$

(Sylinderen har volum $2\pi\sqrt{3}$ og kula $\frac{32}{3}\pi$.)