2.7.12 By about what percentage will the edge length of an ice cube decrease if the cube loses 6% of its volume by melting.

Solution: Let s be its side length. Then the volume is

$$V = V(s) = s^3 \quad \text{and} \quad dV = 3s^2 ds.$$

We are given that the percentage change in V, $100\frac{dV}{V}$, is -0.06. Thus the percentage change in s is about

$$100\frac{ds}{s} = 100\frac{dV/3s^2}{s}$$

$$= \frac{1}{3}100\frac{dV}{s^3}$$

$$= \frac{1}{3}100\frac{dV}{V}$$

$$= -0.02$$

2.7.21 The volume (in liters) of a water tank t min after it starts draining is

$$V(t) = 350(20-t)^2$$

a) How fast is the water draining out after 5 min? 15 min?

b) What is the average rate at which water is draining out during the time interval from 5 to 15 min?

Solution:

a)
$$V'(t) = 350 \cdot 2(20-t)(-1)$$
$$= -700(20-t)$$

The water is draining out in a rate of $|V'(5)| = 10500\ \ell/\text{min}$ after 5 min, and in a rate of $|V'(15)| = 3500\ \ell/\text{min}$ after 15 min.

b) The average rate in the interval [5, 15] is

$$\frac{V(15)-V(5)}{15-5} = \frac{350(5)^2 - 350 \cdot 15^2}{15-5}$$

$$= 350 \frac{5^2 - 15^2}{15-5}$$

$$= 350(5+15)$$

$$= -7500$$

So the water is draining out in an average rate of $7500\ \ell/\text{min}$ in the interval [5, 15]

2.7.23 ~~It's given~~ Solution:

It's given that

$$F(r) = \frac{k}{r^2} \quad \text{and} \quad -1 = F'(4000).$$

We want to find $F'(8000)$:

$$F'(r) = -\frac{2k}{r^3}$$

$$-1 = F'(4000) = -\frac{2k}{(4000)^3} \Rightarrow k = \frac{4000^3}{2}$$

$$\Rightarrow F'(8000) = -\frac{2k}{(8000)^3}$$

$$= -\frac{4000^3}{8000^3}$$

$$= -\left(\frac{4000}{8000}\right)^3$$

$$= -\left(\frac{1}{2}\right)^3 = -\frac{1}{8}$$

so F decreases with a rate of $-\frac{1}{8}$ $\frac{\text{pound}}{\text{mile}}$

2.8:4 Show that $\cos x > 1 - \frac{x^2}{2}$ for $x > 0$ by using the mean-value theorem.

Solution:

Let $f(x) = \cos x + \frac{x^2}{2}$ and let $x > 0$.

By the MVT it exists a $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = \frac{\cos x + \frac{x^2}{2} - 1}{x}$$

$$\overset{MVT}{=} f'(c)$$

$$= -\sin x + x \Big|_{x=c}$$

$$= -\sin c + c$$

$$\overset{Ex2}{>} 0.$$

Multiplying this inequality by the positive number x, we get

$$\cos x > 1 - \frac{x^2}{2} \qquad (1)$$

Claim: (1) also holds for $x < 0$.

proof Let $x < 0$. Then

$$\cos x = \cos(-x) \qquad \text{since cos is even}$$

$$> 1 - \frac{(-x)^2}{2} \qquad \text{by (1) since } -x > 0$$

$$= 1 - \frac{x^2}{2}$$

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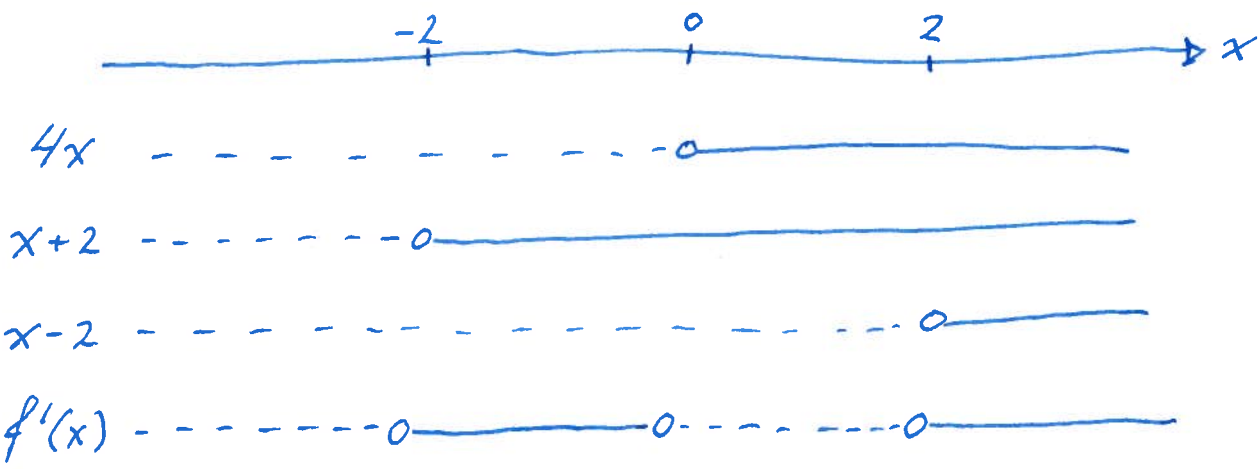
2.8:15 Find the intervals of increase and decrease of the function

$$f(x) = (x^2 - 4)^2$$

Solution: f is differentiable on $\mathbb{R}$ (and thus continuous on $\mathbb{R}$ by THM 1) so by THM 12, f is increasing [decreasing] on the intervals where $f'(x) > 0$ [$f'(x) < 0$].

$$f'(x) = 2(x^2-4)2x = 4x(x+2)(x-2)$$



So f is increasing on $(-2, 0)$ and $(2, \infty)$, and f is decreasing on $(-\infty, -2)$ and $(0, 2)$.

2.9:5 Find dy/dx in terms of x and y when

$$x^2y^3 = 2x - y \qquad (1)$$

Solution (1) is an equation for a curve in the plane. On the parts of the curve where the curve is not vertical, the curve is a graph of a function $y = y(x)$. On those parts, we must have

$$\frac{d}{dx}\, x^2y^3(x) = \frac{d}{dx}\big(2x - y(x)\big)$$

$$\Rightarrow \quad 2xy^3(x) + x^2 3y^2(x)\, y'(x) = 2 - y'(x)$$

$$\Rightarrow \quad y'(x)\big(3x^2y^2(x) + 1\big) = 2\big(1 - xy^3(x)\big)$$

when defined

$$\Rightarrow \qquad \frac{dy}{dx} = y'(x) = 2\,\frac{1 - xy^3}{3x^2y^2 + 1}$$

2.9:16 Find an equation of the tangent to the given curve at the given point.

$$\cos\frac{\pi y}{x} = \frac{x^2}{y} - \frac{17}{2} \quad \text{at } (3,1)$$

Solution: First we check that (3, 1) is a point on the curve:

$$\cos\frac{\pi y}{x} - \frac{x^2}{y} + \frac{17}{2}\Big|_{(x,y)=(3,1)}$$

$$= \cos\frac{\pi}{3} - 9 + \frac{17}{2} = \tfrac{1}{2} - \frac{18}{2} + \frac{17}{2} = 0 \text{ ok!}$$

Implicit diff.:

$$0 = \frac{d}{dx}\left(\cos\frac{\pi y(x)}{x} - \frac{x^2}{y(x)} + \frac{17}{2}\right)\Big|_{(3,1)}$$

$$= -\sin\left(\frac{\pi y(x)}{x}\right)\cdot\pi\,\frac{y'(x)x - y(x)}{x^2} - \frac{2xy(x) - x^2y'(x)}{y^2(x)}\Big|_{(3,1)}$$

$$= -\sin\left(\frac{\pi}{3}\right)\cdot\pi\,\frac{y'(3)\cdot 3 - 1}{9} - \frac{2\cdot3\cdot1 - 9y'(3)}{1^2}$$

$$= -\frac{\sqrt{3}\pi}{18}\left(3y'(3) - 1\right) - 6 + 9y'(3)$$

$$= y'(3)\left(9 - \frac{\sqrt{3}\pi}{6}\right) - 6 + \frac{\sqrt{3}\pi}{18}$$

$$\Rightarrow \quad y'(3) = \frac{6 - \frac{\sqrt{3}\pi}{18}}{9 - \frac{\sqrt{3}\pi}{6}} = \frac{18\cdot6 - \sqrt{3}\pi}{18\cdot9 - 3\sqrt{3}\pi}$$

The tangent at $(3,1)$ has equation

$$T(x) = y'(3)(x-x_0) + y_0$$

$$= \frac{18\cdot 6 - \sqrt{3}\pi}{18\cdot 9 - 3\sqrt{3}\pi}(x-3) + 1$$

2.9:17 Find y'' in terms of x and y.

$$xy = x + y$$

Solution: Assume $y = y(x)$. Then

$$0 = \frac{d}{dx}\left(xy(x) - x - y(x)\right)$$

$$= y(x) + xy'(x) - 1 - y'(x)$$

$$= y'(x)(x-1) + y(x) - 1 \quad \Leftrightarrow \quad y'(x) = \frac{1-y(x)}{x-1} \quad x \neq 1$$

$$\Rightarrow \quad 0 = \frac{d}{dx}\left(y'(x)(x-1) + y(x) - 1\right)$$

$$= y''(x)(x-1) + y'(x) + y'(x)$$

$$= y''(x)(x-1) + 2\,\frac{1-y(x)}{x-1}$$

$$\Rightarrow \quad y''(x) = 2\,\frac{y(x)-1}{(x-1)^2} \quad x \neq 1$$

2.10:12

$$\int \frac{6(x-1)}{x^{4/3}}\,dx = 6\int\left(\frac{x}{x^{4/3}} - \frac{1}{x^{4/3}}\right)dx$$

$$= 6\int\left(x^{-1/3} - x^{-4/3}\right)dx$$

$$= 6\left(\frac{x^{2/3}}{2/3} - \frac{x^{-1/3}}{-1/3}\right) + C$$

$$= 18\left(\frac{x^{2/3}}{2} + \frac{1}{x^{1/3}}\right) + C$$

2.10:26

$$\int \sin^2 x\,dx = -\frac{1}{2}\int\left(1 - 2\sin^2 x - 1\right)dx$$

$$= -\frac{1}{2}\int\left(\cos 2x - 1\right)dx$$

$$= -\frac{1}{2}\left(\frac{1}{2}\sin 2x - x\right) + C$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

2.10.42

IVP: $y'' = x + \sin x$

$y(0) = 2$

$y'(0) = 0$

Solution:

$$y'(x) = \int (x + \sin x)\,dx = \frac{1}{2}x^2 - \cos x + C_1$$

$$0 = y'(0) = -1 + C_1 \quad \Rightarrow \quad C_1 = 1$$

$$y(x) = \int \left(\frac{1}{2}x^2 - \cos x + 1\right)dx$$

$$= \frac{1}{6}x^3 - \sin x + x + C_2$$

$$2 = y(0) = C_2$$

$$\Rightarrow \quad y(x) = \frac{1}{6}x^3 - \sin x + x + 2$$