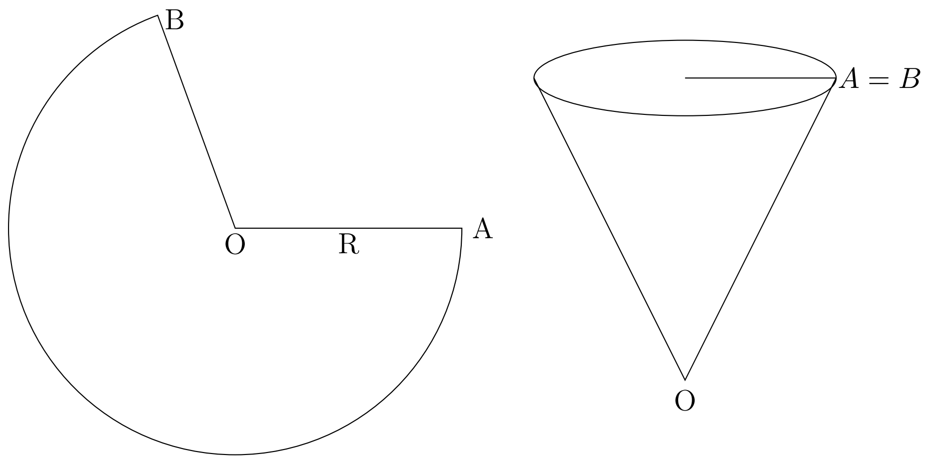
Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4100 Calculus 1
Fall 2013

Exercise set 6
Week 40 (30 September-4 October)

Problem 4.9.48 in Adams A sector is cut out of a disk of radius R, as shown below. The remaining part of the disk is bent up so that the two edges join and a cone is formed.



What is the largest possible volume for the cone?

L'Hôpital's Rule The limit

$$\lim_{x\to 0}\left(\frac{1}{\sin(px)} - \frac{1}{e^{x/2}-1}\right)$$

exists for exactly one real-valued p. Find this value, and calculate the limit.

Convex Functions Adams defines a concave up, or convex, function on an interval I to be a differentiable function such that the derivative f' is an increasing function on I. In this problem, we will show that for differentiable functions this is equivalent to the more general definition of a convex function. We will prove the following:

Assume f is differentiable on an interval I. Then f' is an increasing function on I if and only if for all $x, y \in I$ and all $t \in [0,1]$ we have that

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y).$$

a) First make the substitution $x_1 = x$, $x_3 = y$ and $x_2 = (1-t)x_1 + tx_3$ to prove that if the inequality above holds, then for any x_1, x_2, x_3 in I such that $x_1 < x_2 < x_3$, we have

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2}.$$

HINT: You will need to solve for t in the substitution.

b) Now assume that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2}.$$

holds for $x_1 < x_2 < x_3$. Show that this implies that the inequality

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y).$$

holds for $x, y \in I$ and all $t \in [0,1]$.

c) Now suppose f' increases on I. Use the result from a) and b), and the Mean Value theorem to conclude that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2},$$

and equivalently,

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y).$$

d) For the other implication, suppose that for $x_1 < x_2 < x_3$, we have

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2}.$$

Prove that f' is increasing on I, and that the theorem holds.

Problem 4.7.2 in Adams

Computers can't be trusted blindly.
You can trust mathematics.
-Adams

a) Let $f(x) = x^{10}$. Use Maple or other software to plot $f(x) - \sqrt{f(x)^2}$ on the intervals $(-10, 10)$ and $(-\infty, \infty)$. Why does the plot indicate that the function is not identically equal to 0?

b) Assume that $x > 0$, and use Maple to calculate different values of $f(x)-\sqrt{f(x)^2}$. Also use Maple to simplify the expression. Why does this answer differ from the one indicated in a)?

c) Use the function $g(x) = 2^x \ln(1 + 2^{-x})$ to determine the machine epsilon for your computer. What does the answer tell you? How does this relate to point a)?