ForNotat M1 kap 6.1 del 1

Hvor er vi i pensum?

- Se pdf
- Hva handler kap 6 om?

Integration by parts

- Delvis integrasjon
- Hvorfor?

- Mål:
  - Kjenne igjen når man bruker Int. by parts
  - Kunne utføre Int. by parts

Formel

$$\int u \cdot v' = uv - \int u'v$$

ex: formelen stemmer

$u = x \quad , v = x^2 \quad , u' = 1 \quad , v' = 2x$

$V.S. = \int x \cdot 2x = \int 2x^2 = \frac{2}{3}x^3$

$H.S. = x \cdot x^2 - \int 1 \cdot x^2 = x^3 - \int x^2 = x^3 - \frac{1}{3}x^3 = \frac{2}{3}x^3$

$V.S. = H.S.$

ForNotat M1 kap 6.1 del 2

EXH. 6.1

Oppg 6.1.14

Finn $\int x e^{\sqrt{x}}$

Løsning

For Notat kap 6.1 del 3

Generell formulering

$$\frac{d}{dx} u \cdot v = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

$$(uv)' = u'v + uv'$$

$$\int (uv)' = \int u'v + \int uv' = uv$$

$$\int uv' = uv - \int u'v$$

$\boxed{\text{I boka}}$

Skrivemåte 2:

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v \, dx$$

$$\int u \, dv = uv - \int v \, du$$

Med integrasjonsgrenser

$$\int_0^{\frac{1}{2}} \sin^{-1}(x) \, dx = \int_0^{\frac{1}{2}} 1 \cdot \sin^{-1}(x) dx = \left[x \sin^{-1}(x)\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \sin^{-1}\left(\frac{1}{2}\right) - 0 \cdot 0 + \left[\sqrt{1-x^2}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{2} \frac{\pi}{6} + \sqrt{1-\frac{1}{4}} - \sqrt{1-0} = \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}$$

FORNotat M1 kap 6.2 del 1

Partial fractions

- kap 6.2 del 2
- Mål

  Ta polynombrøker $\frac{P}{Q}$ med $\deg P < \deg Q$ og skrive som en sum av mindre kompliserte brøker
- Hvorfor?

Note
$x^2-1=(x+1)(x-1)$

Eksempel (enkelt)

Skriv $\frac{1}{(x^2-1)}$ som $\frac{A_1}{x+1}+\frac{A_2}{x-1}$

$$\frac{1}{x^2-1} \equiv \frac{A_1}{x+1}+\frac{A_2}{x-1} \qquad \Big| \cdot (x^2-1)$$

$$\frac{x^2-1}{x^2-1} \equiv \frac{A_1}{x+1}(x+1)(x-1) + \frac{A_2}{x-1}(x+1)(x-1)$$

$$1 \equiv A_1(x-1) + A_2(x+1)$$

$$1 \equiv A_1x - A_1 + A_2x + A_2$$

$$1 \equiv (A_1+A_2)x + (-A_1+A_2)$$

$$\Leftrightarrow \begin{cases} A_1+A_2=0 \\ -A_1+A_2=1 \end{cases} \Rightarrow$$

$$A_1=-A_2$$

$$+A_2+A_2=1$$

$$A_2=\frac{1}{2}$$

$$A_1=-\frac{1}{2}$$

FOR Notater M1 kap 6.2 del 2

Altså
$$\frac{1}{x^2-1} \equiv \frac{-1/2}{x+1} + \frac{1/2}{x-1}$$

Generell metode for $\frac{P}{Q}$

i) Sørg for at $\deg P < \deg Q$

ii) Faktoriser Q til lineære & 2. grads

iii) Skriv $$\frac{P}{Q} \equiv \frac{A}{x-a} + \frac{B}{x-b} + \frac{Cx+D}{x^2+1} + \frac{E}{(x-a)^2} + \dots$$

iv) Gjør triksete ting for å finne likninger for $A, B, C, D \dots$

v) Løs likningene

vi) Skriv opp resultatet $\frac{P}{Q} = \dots$ og sjekk at det stemmer ved å legge sammen brøkene

vii) Spis en sjokolade.

Vanskelig eksempel

Use partial fractions on

$$\frac{1}{x^4 - 3x^3}$$

Relatert oppg 6.2.25 : Integrer denne

Løsning

$$\frac{1}{x^3(x-3)} \equiv \frac{A}{x-3} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3}$$

Hvor mange?

Trenger likninger :

$\cdot(x-3)$ sett inn $x = 3$

$$\frac{1\cdot(x-3)}{x^3\cdot(x-3)} = \frac{A(x-3)}{x-3} + \frac{B(x-3)}{x} + \frac{C}{x^2}(x-3) + \frac{D}{x^3}(x-3)$$

Egentlig $\lim_{x \to 3}$

$x = 3 \Rightarrow$

$$\frac{1}{3^3} = A + 0 + 0 + 0 = A$$

Ny likning :

$\cdot(x^3)$ sett inn $x = 0$

$$\frac{1}{0-3} = 0 + 0 + 0 + D \Rightarrow D = -\frac{1}{3}$$

Den store likningen :

Gang med $x^3(x-3)$

$$1 \equiv Ax^3 + Bx^2(x-3) + Cx(x-3) + D(x-3)$$

$$1 \equiv x^3(A+B) + x^2(-3B+C) + x(-3C+D) - 3D$$

Dette gir

$A + B = 0$

$-3B + C = 0$

$-3C + D = 0$

$-3D = 1$

Fra triks

$A = \frac{1}{27}$

$D = -\frac{1}{3}$

Så:

$B = -\frac{1}{27}$, $C = 3B = -\frac{1}{9}$, $D = -\frac{1}{3}$, $A = \frac{1}{27}$

Altså

$$\frac{1}{x^3(x-3)} \equiv \frac{\frac{1}{27}}{x-3} - \frac{1/27}{x} - \frac{1/9}{x^2} - \frac{1/3}{x^3}$$

Merk !

$$\frac{1}{x^3(x-3)} \equiv \frac{1/27}{x-3} - \frac{\frac{1}{27}x^2 + \frac{1}{9}x + \frac{1}{3}}{x^3}$$

Integrer en brøk

$$I = \int \frac{P(x)}{Q(x)} \quad \boxed{\text{polynomer}} \qquad = \int \frac{x^3 - x + 1}{x^2 - 1}$$

$$= \int \frac{P_2(x)}{Q(x)} + R(x) \quad \boxed{\deg P_2 < \deg Q} \qquad = \int \frac{1}{x^2 - 1} + x$$

$$= \int \frac{A}{x-a} + \frac{B}{x-b} + \dots + \int R(x) \qquad = \int \frac{-1/2}{x+1} + \frac{1/2}{x-1} + x$$

$$= \text{Sum av disse} \begin{cases} \int \frac{x}{x^2 + a^2} = \frac{1}{2} \ln(x^2 + a^2) + C \\ \int \frac{1}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C & a \geq 0 \\ \int \frac{1}{x-a} = \ln|x-a| + C & a < 0 \\ \int \frac{1}{(x-a)^N} = \frac{1}{-N+1} (x-a)^{-N+1} & N \geq 2 \end{cases}$$

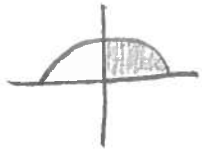
Inverse Substitutions

- Vinkelsubstitusjoner
- Mål

  Kjenne igjen & Utføre "Inverse Substitutions"

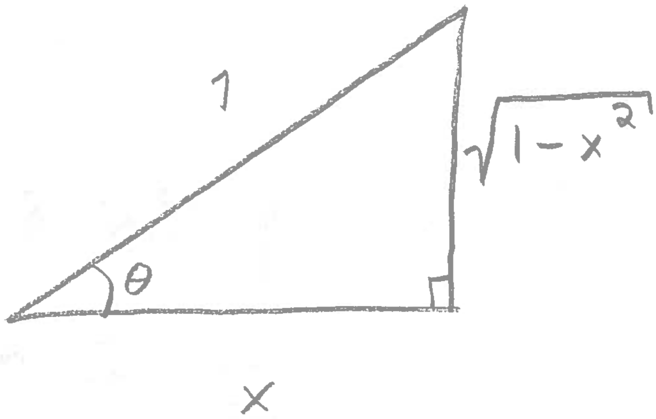
Enkelt Eksempel

Finn $I = \int_0^1 \sqrt{1-x^2}\,dx$



Løsning:

"Hmm, ser ut som sirkel, derivert av arcsin/cos/tan"



$\sin\theta = \sqrt{1-x^2}$

$\frac{d}{dx}\sin\theta = \frac{d}{dx}\sqrt{1-x^2}$

$$\cos\theta \cdot \frac{d\theta}{dx} = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)$$

$$\cos\theta \cdot \frac{d\theta}{dx} = \frac{-x}{\sin\theta} = \frac{-\cos\theta}{\sin\theta}$$

$$\frac{d\theta}{dx} = \frac{-1}{\sin\theta} \Rightarrow dx = -\sin\theta\, d\theta$$

$$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$$

$$I = \int_{x=0}^{x=1} \sin\theta \cdot (-\sin\theta)\, d\theta$$

$$I = -\frac{1}{2}\int_{x=0}^{1} 1 - \cos 2\theta\, d\theta = -\frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{x=0}^{1}$$

FORNOTM1
kap 6.3 del 2

...

$$I = -\frac{1}{2}\left[\arccos x - \frac{1}{2}\sin(2\cdot\arccos(x))\right]_0^1$$

$$= -\frac{1}{2}\left(0 - \frac{1}{2}\sin(2\cdot 0)\right) + \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}\sin\left(2\cdot\frac{\pi}{2}\right)\right)$$

$$= 0 + \frac{\pi}{4} - 0$$

$$= \underline{\underline{\frac{\pi}{4}}}$$

Generell diskusjon

- Må tegne!
  - bruk tegningen hele tiden
- Må kunne/finne $\sin^N\theta$ etc formler
- Kan være vanskelig å se om det vil funke
  - Som andre teknikker

Vanskelig eksempel

EX Oppgave 6.3.10

Finn $\displaystyle\int \frac{\sqrt{9+x^2}}{x^4}\,dx$

(Ta nytt ark)

Kap 6.4 Other Methods

- Mål
  - Ingen
- Gjette m/ubest. koeff
  " $\int xe^x$ var $axe^x + be^x$ for noen tall a, b.
- Maple etc.
- Numerikk → kap 6.6 + 6.7

Improper integrals

- Vekte integraler
- Mål

  Kunne gi mening til flere typer integraler
- Hvorfor?

Enkelt eksempel

Finn $\int_{-1}^{1} \frac{1}{x^2}\,dx$

Løsning

$$I = \int_{-1}^{1} \frac{1}{x^2}\,dx$$ stykkevis kontinuerlig

$$I = \int_{0}^{1} \frac{1}{x^2}\,dx + \int_{-1}^{0} \frac{1}{x^2}\,dx = 2\int_{0}^{1} \frac{1}{x^2}\,dx$$

kontinuerlig, men ikke på intervallet $[0,1]$

$$I = \lim_{\varepsilon \to 0^+} 2\int_{\varepsilon}^{1} \frac{1}{x^2}$$ kontinuerlig!!

$$I = \lim_{\varepsilon \to 0^+} 2\left[\frac{-1}{x}\right]_{\varepsilon}^{1} = \lim_{\varepsilon \to 0^+} 2\cdot\left(-1 + \frac{1}{\varepsilon}\right)$$

$$= 2\lim_{w \to \infty}(-1 + w) = 2\cdot\infty = \underline{\underline{\infty}}$$

Generelt

Hvis ting går mot ∞, bytt ut med grenseverdier

$$\int_a^\infty f(x) \overset{\text{def}}{=} \lim_{R\to\infty} \int_a^R f(x)\,dx$$

Hva kan gå galt?

Ikke konvergens:

$$\lim_{R\to\infty} \int_0^R \sin\theta\, d\theta = \lim_{R\to\infty} \cos(R) - \cos(0) = \text{ikke def.}$$

Tosidig galskap

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty} \int_0^R x\,dx + \lim_{S\to\infty} \int_{-S}^0 x\,dx$$

$$= \lim_{R\to\infty} \frac{1}{2}R^2 + \lim_{S\to\infty} (0 - S^2)$$

$$= \infty - \infty$$

Kan dette fikses med $\int_{-R}^{R} x\,dx = 0$?

– Nei!

Vi sier at $\int_0^\infty \sin\theta$ og $\int_{-\infty}^{\infty} x$ ikke er integrerbart.

Vanskelig eksempel

EX: Prove Thm 2b $\approx$ oppg 6.5.27

Vis at

$$\int_0^a x^{-p}\,dx = \begin{cases} \dfrac{a^{1-p}}{1-p} & \text{if } p<1 \\ \infty & \text{if } p\geq 1 \end{cases}$$

Løsning

For Notat M1 kap 6.1-6.5

Appendix O

6.1.14

Kladd EX 6.1:

$\int x e^{\sqrt{x}} dx = x \int e^{\sqrt{x}} - \int 1 \cdot \int e^{\sqrt{x}}\, dx\, dx$ not poss.

$\int u^2 e^u \, 2u\, du$ $\quad \frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u}$ $\quad \cancel{= \frac{1}{2}\int u e^u = \frac{1}{2} u e^u - \frac{1}{2}\int 1 \cdot e^u}$

Skriv v & w $\quad \cancel{= \frac{1}{2} u e^u - \frac{1}{2} e^u = \frac{1}{2}(u-1)e^u = \frac{1}{2}(\sqrt{x}-1)e^{\sqrt{x}}}$

$\cancel{\text{deriv } \frac{1}{2}\frac{1}{2\sqrt{x}} e^{\sqrt{x}} + \frac{1}{2}(\sqrt{x}-1)\sqrt{x} e^{\sqrt{x}} = e^{\sqrt{x}}}$

$$I = \int \underset{v}{2u^3} \underset{w'}{e^u} du = \underset{v\,w}{2u^3 e^u} - \int \underset{v'w}{6u^2 e^u} du$$

$$\int \underset{v}{u^2} \underset{w'}{e^u} du = \underset{v}{u^2} \underset{w}{e^u} - \int \underset{v'\,w}{2u e^u} du$$

$$= u^2 e^u - \left(2ue^u - \int 2e^u du\right)$$

$$= u^2 e^u - 2ue^u + 2e^u$$

$$I = 2u^3 e^u - 6u^2 e^u + 12ue^u - 12e^u$$

$$= \left(e^{\sqrt{x}}\right)\left(2x^{\frac{3}{2}} - 6x + 12x^{\frac{1}{2}} - 12\right)$$

D.H. W.Alpha: OK!

$\int_\varepsilon^a x^{-p} = \frac{x^{-p+1}}{-p+1}$ for $p \neq 0$ $\quad \boxed{\text{case 2 } p = 1}$

$\frac{a^{-p+1} - \varepsilon^{-p+1}}{-p+1}$ $\quad$ case 3 $\lim_{\varepsilon \to 0^+}$

$\varepsilon \to 0$ så vil $\varepsilon^{-p+1} \to 0$ $p < 1$, ∞ $p > 1$

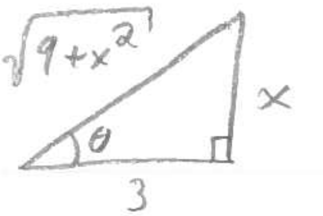
$\to \frac{a^{-p+1}}{-p+1}$ $p < 1$

$\to \infty$ itk pos. & integrand

$p = 0 \Rightarrow \int_0^a x^{-1} = \ln(a) - \ln\varepsilon$, $\varepsilon \to 0$, $\ln \varepsilon \to -\infty$

6.3.10

$$\int \frac{\sqrt{9+x^2}}{x^4} dx$$



$\cancel{\tan\theta}$

$\cos\theta = \frac{3}{\sqrt{9+x^2}}$

$\sin\theta = \frac{x}{\sqrt{9+x^2}}$

$\tan\theta = \frac{x}{3} = \frac{\sin\theta}{\cos\theta}$

$\frac{1}{\cos^2\theta} \cdot \frac{d\theta}{dx} = \frac{1}{3}$

$$= \int \frac{3}{\cos\theta} \cdot \frac{\cancel{3^4} \cos^4\theta}{3^4 \sin^4\theta} \cdot \frac{3}{\cos^2\theta} d\theta$$

$$= \int \frac{1}{3^2} \cdot \frac{\cos}{\sin^4} d\theta$$

$$= \int \frac{1}{9} u^{-4} du = \frac{1}{9}\frac{1}{-3} u^{-3}$$

$$= \frac{-1}{27} \sin^{-3}\theta = \frac{-1}{27} \frac{\sqrt{9+x^2}^{\,3}}{x^3}$$

deriv $\frac{\frac{3}{2}\sqrt{9+x^2} \cdot 2x \cdot x^3 - \sqrt{9+x^2}^{\,3}\, 3x^2}{x^6}$

$\to \frac{1}{x^4}\left(\sqrt{9+x^2}\right)\left(\underbrace{3x^2 - (27 + 3x^2)}_{\text{should be } -27}\right)$ ok!

Appendix del 1

Ekstra eksempel A.1

Finn $\int e^x \sin x$

Løsning

$$I = \int \underset{v'}{e^x} \underset{w}{\sin x} = \sin x \, e^x - \int \underset{v'}{e^x} \underset{w}{\cos x}$$

$$= \sin x \, e^x - \left(e^x \cos x - \int e^x \cdot (-\sin x) \right)$$

$$= e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x}_{= I}$$

$$2I = e^x \sin x - e^x \cos x$$

$$I = \frac{1}{2} e^x (\sin x - \cos x) \quad + C$$