Geometric Integration and Lie group methods, THREAD
Network-wide training event 6

February 8, 2021

FIFTH EXERCISE. This is the second exercise of the second part of the course. Our aim in this exercise is to derive properties of the Lie group $SE(3)$ and its Lie algebra $\mathfrak{se}(3)$. If you will be using this Lie group in modeling and simulation later, you may want to take some short cuts by using directly that this Lie group can be realised as a matrix group. Still, it may be useful one time to study its properties purely in the general framework of Lie groups and Lie algebras. So the purpose and learning objective of this exercise is to use the general theory discussed in the lectures to derive properties for this particular example of a Lie group, and we shall give you some assistance on the way. To get the best learning outcome, it may be a good idea to look a couple of minutes at the general theory from the lectures before diving into each problem.

The exercise may appear to be long, and perhaps you cannot complete everything in 3 hours. But just do as much as you can, we will "approve" everything that seems to be a result of some hours of hard work.

Preliminaries about $SE(3)$. This group is usually constructed as a semidirect product between $SO(3)$ and $\mathbb{R}^3$, sometimes you will see the notation $SE(3) = SO(3) \rtimes \mathbb{R}^3$. It consists of pairs that we shall often denote (g, u) where $g \in SO(3)$ and $u \in \mathbb{R}^3$. The meaning of "semidirect product" as opposed to "direct product" is for us here just that the product structure is coupling the two components, but you need not worry about this for the moment. The important thing in this exercise is that the product is defined as

$$(g, u) \cdot (h, v) = (gh, gv + u)$$

where the first component is a multiplication between two $SO(3)$-matrices and the second component is a matrix-vector product and a sum of two vectors. It is easy then also to check that the identity element is $e = (I, 0)$ the pair consisting of the 3×3 identity matrix at the 0-vector with three components. The inverse of an element is $(g, u)^{-1} = (g^{-1}, -g^{-1}u) = (g^T, -g^T u)$.

The Lie algebra $\mathfrak{se}(3)$ can be represented as pairs $(\hat{\xi}, \eta)$ where $\hat{\xi} \in \mathfrak{so}(3)$ (skew-symmetric 3×3 matrices) and $\eta \in \mathbb{R}^3$. Whenever we can, we will represent $\hat{\xi}$ as the vector in $\mathbb{R}^3$ that defines $\hat{\xi}$ via the hat map. A useful thing to remember is that $\hat{\xi}\eta = \xi \times \eta$ where $\times$ is the cross product between vectors in $\mathbb{R}^3$.

Problem 1. Find the left and right invariant vector fields on $SE(3)$. Here the intention is that you take an element $(\xi, \eta) \in T_eSE(3) \equiv \mathfrak{se}(3)$ such that the left invariant vector field which we denote $X_{(\xi,\eta)}$ equals (ξ, η) at $e = (I, 0)$. It can the be computed by left translation

$$X_{(\xi,\eta)}\big|_{(g,u)} = TL_{(g,u)}(\xi, \eta)$$

Then find in a similar way the right invariant vector fields on $SE(3)$.

Problem 2. The exponential, $\exp : \mathfrak{se}(3) \to SE(3)$. For the learning experience, one could calculate the exponential map both as the flow of left and right invariant vector fields respectively. Start by left invariant case, to calculate $\exp(\xi, \eta)$ consider

$$(\dot{g}(t), \dot{u}(t)) = X_{(\xi,\eta)}\big|_{(g(t),u(t))}, \qquad (g(0), u(0)) = (I, 0)$$

Substitute what you found for $X_{(\xi,\eta)}$ and solve for each of the two components $g(t)$ and $u(t)$. Evaluate $\exp(\xi, \eta) = (g(1), u(1))$. If you are doing well with time, repeat for the right invariant vector field to check that you get the same result.

Problem 3. Adjoint and coadjoint representations. The map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is generally defined for any $g \in G$ as $\mathrm{Ad}_g(h) = TL_g \circ TR_{g^{-1}}(\xi)$, $\xi \in \mathfrak{g}$. It is a left Lie group action on the space $\mathfrak{g}$ meaning that $\mathrm{Ad}_{gh} = Ad_g \circ \mathrm{Ad}_h$. In the case we consider, we look at $\mathrm{Ad}_{(g,u)}(\xi, \eta)$ where $(g, u) \in SE(3)$ and $(\xi, \eta) \in \mathfrak{se}(3)$. Using the expressions you found for $TL_{(g,u)}$ and $TR_{(g,u)}$ in Problem 1, find an expression for

$$\mathrm{Ad}_{(g,u)}(\xi, \eta)$$

Let us now consider the coadjoint representation, defined as $\mathrm{Ad}^*_{(g,u)^{-1}}$, note that we have now replaced (g, u) by $(g, u)^{-1}$ in the subscript. If you need a little more explanation about what the adjoint is and how it is computed, see the section *Additional Information* at the end of the exercise.

We can proceed as follows: Pick a pair $(\mu, \nu) \in \mathfrak{se}(3)^*$, they are both represented as 3-vectors. The pairing with an element $(\xi, \eta) \in \mathfrak{se}(3)$ is just

$$\langle(\mu, \nu), (\xi, \eta)\rangle = \mu^T\xi + \nu^T\eta$$

Now calculate

$$\langle\mathrm{Ad}^*_{(g,u)^{-1}}(\mu, \nu), (\xi, \eta)\rangle = \langle(\mu, \nu), \mathrm{Ad}_{(g,u)^{-1}}(\xi, \eta)\rangle$$

substitute what you found for $\mathrm{Ad}_{(g,u)}$ above, while remembering that you need to replace (g, u) by $(g^{-1}, -g^{-1}u)$. Then reorganise such that you obtain a pairing where the right component is just (ξ, η). What is in the left component is then simply $\mathrm{Ad}^*_{(g,u)^{-1}}(\mu, \nu)$.

Problem 4. The Lie bracket in $\mathfrak{se}(3)$. We shall now use an alternative definition of the Lie bracket to avoid the somewhat cumbersome commutator of left invariant vector fields. Generally, for a Lie group G with Lie algebra $\mathfrak{g}$, we can think of $\mathrm{Ad}_g(\xi)$ as a map from G to $\mathfrak{g}$ (for a fixed ξ). Let us for a moment call it $A_\xi(g)$, $A_\xi : G \to \mathfrak{g}$. A well known fact is that the Lie bracket can be obtained by taking the tangent map, i.e.

$$[\xi, \zeta] = TA_\xi|_e(\zeta).$$

For $\mathfrak{se}(3)$ we then take a curve $(g(t), u(t))$ with $(g(0), u(0)) = (I, 0)$, $(\dot{g}(0), \dot{u}(0)) = (\hat{\xi}_1, \eta_1)$ and we compute

$$[(\xi_1, \eta_1), (\xi_2, \eta_2)] = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{(g(t),u(t))}(\xi_2, \eta_2)$$

Do this calculation to find an expression for the Lie bracket in $\mathfrak{se}(3)$ in terms of the cross products of 3-vectors.

Problem 5. The inifinitesimal generator of the left coadjoint action on $\mathfrak{se}(3)^*$. We now consider the map $\lambda_* : \mathfrak{g} \to \mathcal{X}(\mathfrak{se}(3)^*)$ induced by the left coadjoint action $\Lambda((g, u), (\mu, \nu)) = \mathrm{Ad}^*_{(g,u)^{-1}}(\mu, \nu)$. So taking an element $(\xi, \eta) \in \mathfrak{se}(3)$, λ_* maps it to a vector field on the linear space $\mathfrak{se}(3)^*$. Recall that if the group action is denoted $\Lambda((g, u), (\mu, \nu))$ then we defined the infinitesimal generator to be

$$\lambda_*(\xi, \eta)|_{(\mu,\nu)} = \left.\frac{d}{dt}\right|_{t=0} \Lambda(\exp(t(\xi, \eta)), (\mu, \nu))$$

Rather than working with the somewhat complicated expression for $(g(t), u(t)) := \exp(t(\xi, \eta))$, it suffices to use that this curve passes through the identity at $t = 0$ and that its tangent there is $(\hat{\xi}, \eta)$. Compute

$$\lambda_*(\xi, \eta)|_{\mu,\nu} = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{(g(t),u(t))^{-1}}(\mu, \nu)$$

Problem 6. Finally, let us put it all to use. The heavy top is a well known test problem for Lie group integrators. Following Marsden and Ratiu [1] we can model the heavy top equations on $\mathfrak{se}(3)^*$. This means that the solution consists of a pair $(\mu(t), \nu(t))$ where $\mu(t)$ is the angular momentum of the body in $\mathfrak{so}(3)^*$ and $\nu(t)$ is a vector in $\mathbb{R}^3$. In fact, as in Problem 4, we shall also represent μ as a vector in $\mathbb{R}^3$. The equations are:

$$\begin{aligned} \dot{\mu} &= \mu \times \mathbb{I}^{-1}\mu + \nu \times c\,\chi \\ \dot{\nu} &= \nu \times \mathbb{I}^{-1}\mu \end{aligned}$$

As usual $\mathbb{I}$ is the inertia tensor, and c is a real constant the mass times the gravity constant. v is a vector in the direction from the fixed point of the heavy top to the center of mass (in body coordinates) so this is a vector that does not change over time.

You shall now write the differential equations for the heavy top in the form

$$(\dot{\mu}, \dot{\nu}) = \lambda_*(f(\mu, \nu))|_{(\mu,\nu)}$$

where λ_* is the infinitesimal generator of the left coadjoint action from the previous problem and $f : \mathfrak{se}(3)^* \to \mathfrak{se}(3)$. Find f.

Additional information. In case you are unfamiliar with adjoint operators, we make a couple of remarks on what they are and how you can find them. Suppose more generally that V is a linear space and that $A : V \to V$ is a linear operator. V^* consists of all linear functions on V, so if $\mu \in V^*$ and $v \in V$ we have $\mu(v) = \langle \mu, v \rangle$ a real number. The adjoint of A is called $A^* : V^* \to V^*$ and is defined through

$$\langle A^* \mu, v \rangle = \langle \mu, Av \rangle \quad \text{for all } \mu \in V^* \text{ and } v \in V.$$

With matrices, the adjoint A^* is just the transpose A^T.

References

[1] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1999. A basic exposition of classical mechanical systems.