IMPROPER PRIORS
&
FIDUCIAL INFERENCE

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Abstract

The use of improper priors flourish in applications and is as such a central part of contemporary statistics. Unfortunately, this is most often presented without a theoretical basis:

> *"Improper priors are just limits of proper priors ... "*

We present ingredients in a mathematical theory for statistics which generalize the axioms of Kolmogorov so that improper priors are included. A particular by-product is an elimination of the famous marginalization paradoxes in Bayesian and structural inference. Secondly, we demonstrate that structural and fiducial inference can be formulated naturally in this theory of conditional probability spaces. A particular by-product is then a proof of conditions which ensure coincidence between a Bayesian posterior and the fiducial distribution. The concept of a conditional fiducial model is introduced, and the interpretation of the fiducial distribution is discussed. It is in particular explained that the information given by the prior distribution in Bayesian analysis is replaced by the information given by the fiducial relation in fiducial inference.

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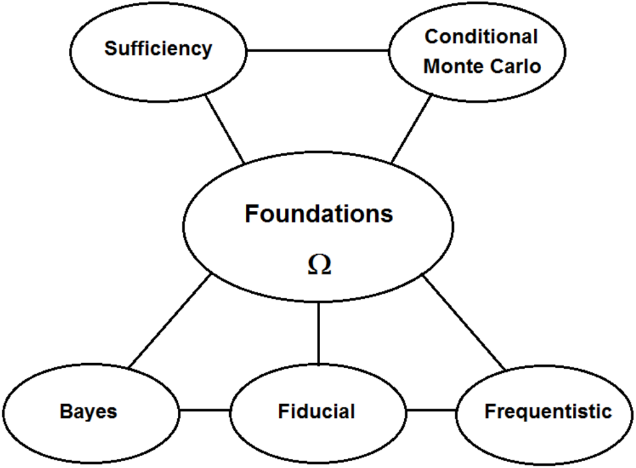
The Large Picture



A motivating problem gives it all

- Initial problem: Generate data $X = \chi(U, \theta)$ conditionally given a sufficient statistic $T = \tau(U, \theta) = t$.
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- Corrected solution: The simulated data must be weighted, and the weight depends on an arbitrarily chosen improper distribution for the parameter (*Biometrika* 2003 & 2005).
- Realization after many years: Oh . . . the resulting unweighted adjusted parameters are fiducial and the weighted are Bayes posterior . . . (*Comm. Stat.* 2015).
- Ingredients: Optimal inference (*Ann. Stat.* 2013), Improper distributions (*Ann. Stat.*, 2010), Sufficiency, Fiducial and Bayesian distribution, Borel paradox, Marginalization paradox, Axioms for statistics (*Comm. Stat.* 2016).

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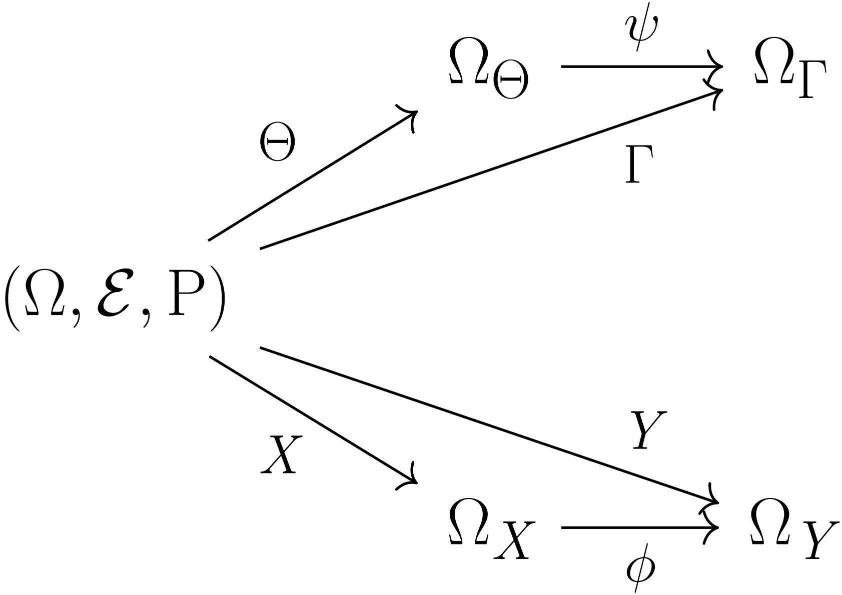
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Ω_Γ
Θ
Γ
$(\Omega, \mathcal{E}, \mathrm{P})$
Y
X
Ω_X
ϕ
Ω_Y

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- The **basic space** Ω is a conditional probability space

(1) $$(\Omega, \mathcal{E}, \mathrm{P})$$

given by axioms that generalize the axioms of Kolmogorov.

- The **model observation** X is a measurable function

(2) $$X : \Omega \to \Omega_X$$

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Conditional probability and more

- Conditional probability and conditional expectation are defined by conditioning on a σ-finite σ-field $\mathcal{F}$ of events.

- If Y is σ-finite, then:

  (A) $(\Omega_Y, \mathcal{E}_Y, \mathrm{P}_Y)$ is a conditional probability space.

  (B) $\mathrm{P}^y(A) = \mathrm{P}(A \mid Y = y)$ is well defined.

- If Θ and X are σ-finite, then **model** $\{\mathrm{P}_X^\theta(A) = \mathrm{P}^\theta(X \in A)\}$ and **posterior** $\{\mathrm{P}_\Theta^x(B) = \mathrm{P}^x(\Theta \in B)\}$ are well defined.

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- A common, but sadly imprecise statement:

- What kind of limit?

- Limit in what space?

- Compare with:

  "Real numbers are just limits of rational numbers."

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- Renyi: The uniform law P_Θ for Θ is defined by

$$\forall n \in \mathbb{N}, \quad (\Theta \mid -n \leq \Theta \leq n) \sim \mathrm{U}[-n, n] \tag{4}$$

- The family of sets $\mathcal{B} = \{[-n, n] \mid n \in \mathbb{N}\}$ is a *bunch* of sets, and the family of conditional probabilities

$$\{\mathrm{P}_\Theta(\cdot \mid B) \mid B \in \mathcal{B}\} \tag{5}$$

defines a conditional probability space.

- The uniform law is not defined by a limit. Consider instead all the conditional uniform laws together as a (new) concept:

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Improper priors and sufficiency

- T is sufficient relatively to X for the parameter θ if $X|(T=t,\Theta=\theta)$ does not depend on θ.
- Equivalently ($\forall$ priors): X, Θ are conditionally independent given T, so $(\Theta \mid X, T) \sim (\Theta \mid T)$. If, additionally, $T = \hat{\tau}(X)$, then $(\Theta \mid X) \sim (\Theta \mid T) \sim$ Bayes posterior.
- Theorem: Sufficiency implies that $[X|(T=t,\Theta=\theta)] \sim [X \mid T=t]$.
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Sufficiency and optimal inference

- Sufficiency principle: Valid inference must be based on T for all sufficient T. It can be disputed. Halmos argument relies on accepting randomized procedures.
- If $\phi(X)$ is an estimator, then $\mathrm{E}^{\theta}(\phi(X) \mid T)$ is an estimator with smaller (or equal) convex risk. If T is complete (and minimal), then it is the unique optimal estimator.
- An exact test for ($H_0 : \alpha = \alpha_0, \theta$ arbitrary) is obtainable if T is sufficient for the nuisance parameter θ. This gives exact confidence distributions. Lehmann: Optimality follows from this with additional assumptions.
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  Challenge: Calculate characteristics of the posterior P_{Θ}^{x}.

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- Bayesian and Fiducial arguments can sometimes be used to obtain excellent (frequentistic) instruments $\varphi = \phi(x)$ beyond the case of $\varphi = \phi(x)$ equal to a confidence distribution (*Ann. Stat.*, 2013).

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- Bayesian and Fiducial arguments can sometimes be used to obtain excellent (frequentistic) instruments $y = \phi(x)$ beyond the case of $y = \phi(x)$ equal to a confidence distribution (*Ann.Stat.*, 2013).

Statistical inference (BFF)

- **Bayes**: Uncertainty P_{Θ}^{x} directly for the particular experiment at hand based on observation x, model P_{X}^{θ}, and prior P_{Θ}.

  Challenge: Calculate characteristics of the posterior P_{Θ}^{x}.

- **Fiducial**: To be discussed on the next slides!

- **Frequentist**: Uncertainty indirectly from properties of the instrument in use based on observation x and model P_{X}^{θ}.

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Simple fiducial inference

- Let $X = \Theta + U$ with $(U \,|\, \Theta = \theta) \sim \mathrm{N}(0, I)$.
- Assume that $x = X(\omega) = (x_1, x_2)$ has been observed.
- Can **You** give a probability judgement regarding the unknown parameter $\theta = \Theta(\omega)$ when ω is unknown?

The simple fiducial argument:

1. **Known:** $x = X(\omega) = \Theta(\omega) + U(\omega)$
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- Let $X = \Theta + U$ with $(U \mid \Theta = \theta) \sim \mathrm{N}(0, I)$.
- Assume that $x = (x_1, x_2)$ has been observed.
- Assume that it is known that $C(\theta) = \theta_1 - \theta_2 = 0$.
- Can **You** give a probability judgement regarding the unknown parameter θ?

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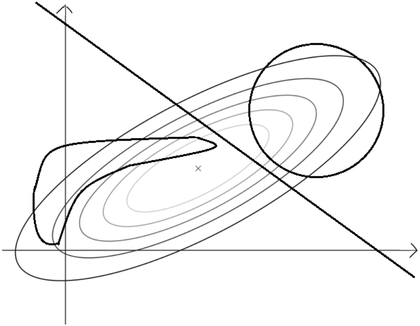

Level curves for a bivariate fiducial together with three possible curves for restriction on the parameter space (Fisher, 1973 ed, p.138).

- A conditional fiducial model (U, χ, C) is given by a fiducial model (U, χ) for the observation X and a condition $C(\Theta) = c$.
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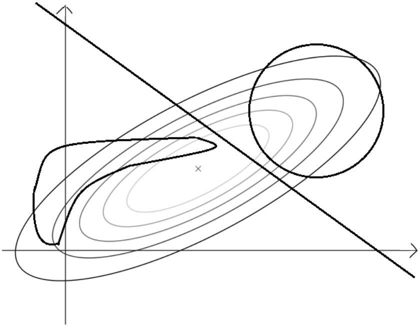

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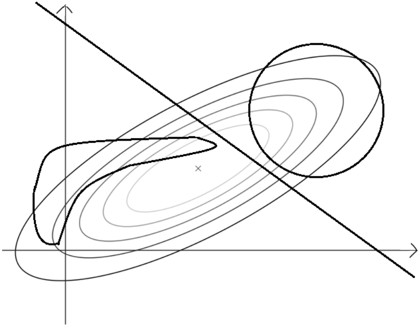

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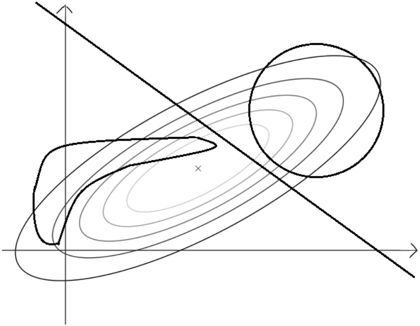

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- Fiducial: $\Theta^x = x[U^x]^{-1}$ with $U^x \sim (U \,|\, \Theta = \theta)$
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$= \mathrm{E}^\theta \, \ell \, [e, \phi(U)]$	Invariance of loss l
$= \mathrm{E}^\theta \, \ell \, [\Theta^x, \Theta^x\phi(U)]$	Invariance of loss l
$= \mathrm{E}^\theta \, \ell \, [\Theta^x, \phi(\Theta^x U)]$	Equivariance of ϕ
$= \mathrm{E}^\theta \, \ell \, [\Theta^x, \phi(x)]$	Fiducial equation

Conclusion: The risk is determined by the fiducial distribution, and an optimal instrument ϕ - if it exists - is determined by the fiducial distribution. There is no need for a Bayes prior in this argument!

Final comments on the involved ideas

- **Intuition: The information in the prior is replaced by the (weaker) information in the fiducial relation.**
- Instead of deciding a prior P_Θ: Decide on a distribution for U^x for a given observation x and the given fiducial relation.
- Fiducial distributions can give more than confidence intervals: Good, possibly optimal, instruments more generally.
- A theory with improper priors have been used repeatedly above. This is useful also more generally. It gives for instance precise limit statements involving priors, and resolves marginalization type of paradoxes.
- In the above arguments the law of $(U \,|\, \Theta = \theta)$ does not depend on θ. Fraser considers interesting models where $(U \,|\, \Theta = \theta)$ has a distribution that depends on θ through a shape parameter.

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