Random walk in dynamic random environment with applications in biology

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PROBABILISTIC STRUCTURES
IN EVOLUTION

DFG SPP 1590

Ancestral lineages and long-time behaviour of population models with interactions

Matthias Birkner (Mainz), Jiri Cerny (Basel), Nina Gantert (TU München) and Andrej Depperschmidt (Nürnberg)

See `https://arxiv.org/abs/1912.02558` for an overview paper of the project by Birkner and Gantert (2019).

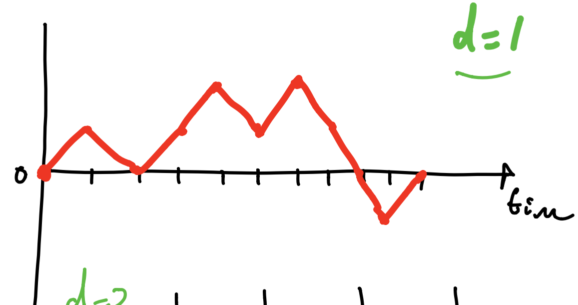
Random Walk

Fundamental class of stochastic processes with a wide range of applications.

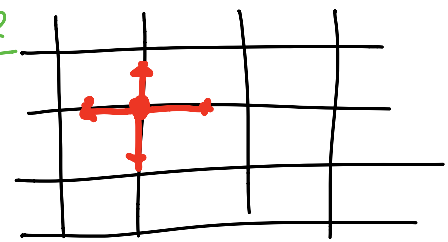
Random walk on $\mathbb{Z}^d$

- $\mathbb{P}(X_0 = 0) = 1$
- $\mathbb{P}(X_n = x \pm e_k \mid X_{n-1} = x) = \frac{1}{2d}$

$d=1$



$d=2$

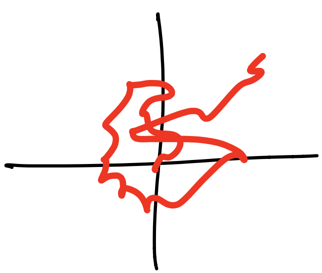


Random Walk

Mathematically, one is often interested in the long term behaviour of (X_n). For the classical models much is known.

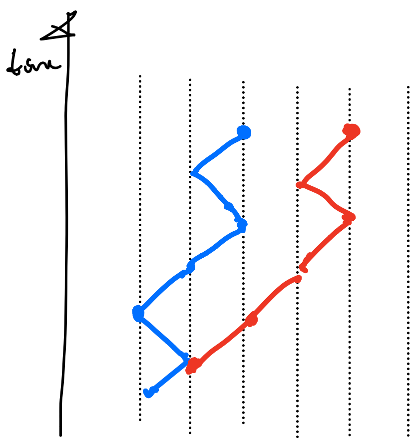
Random Walk on $\mathbb{Z}^d$

- (X_n) is recurrent on $\mathbb{Z}$ and $\mathbb{Z}^2$
- (X_n) is transient on $\mathbb{Z}^d$, $d \geq 3$
- SLLN: $\mathbb{P}\left(\lim_{n\to\infty} \frac{X_n}{n} = 0\right) = 1$
- CLT: $\frac{1}{\sqrt{n}} X_{\lfloor nt \rfloor} \Rightarrow W_t \sim N(0,t)$



Random walk models for ancestral lineages

- In biological applications, random walks are often used to model ancestral lineages in spatial populations



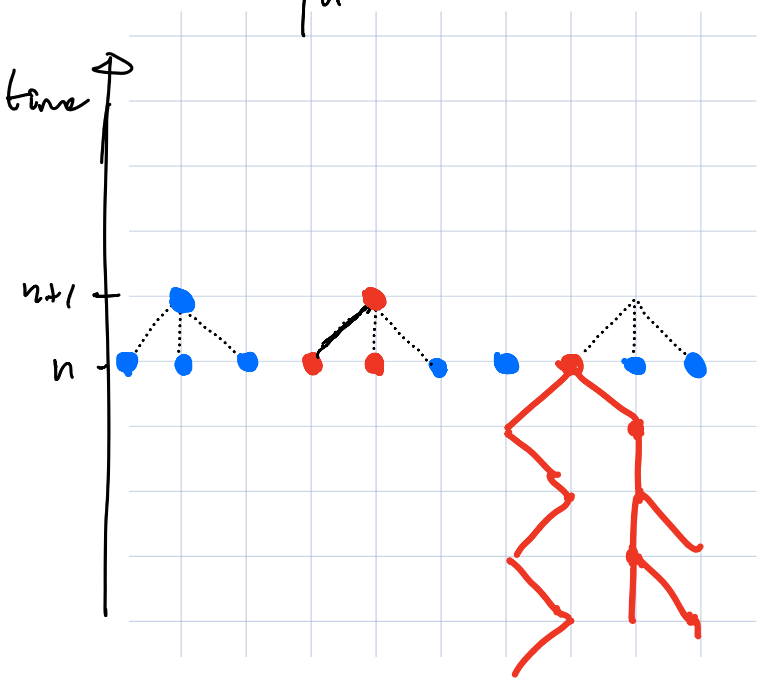
- Do two individuals have the same ancestors?
- Where are the ancestors located?

- Due to fluctuating population sizes and local regulation, these are fairly complicated objects.
- However, via an ad-hoc assumption of constant local population size, these are replaced by simple random walks.

Spatial population models

The discrete-time contact process

Let $\eta_n(x) = $ ● if (x,n) is occupied, otherwise $\eta_n(x) = $ ●



- the dotted lines are "open" with probability $p > 0$ (independently)
- $\eta_{n+1}(y) = $ ● iff there is an open path from x where $\eta_n(x) = $ ●
- If $p > p_c$, then the population survives forever

The discrete-time Contact Process

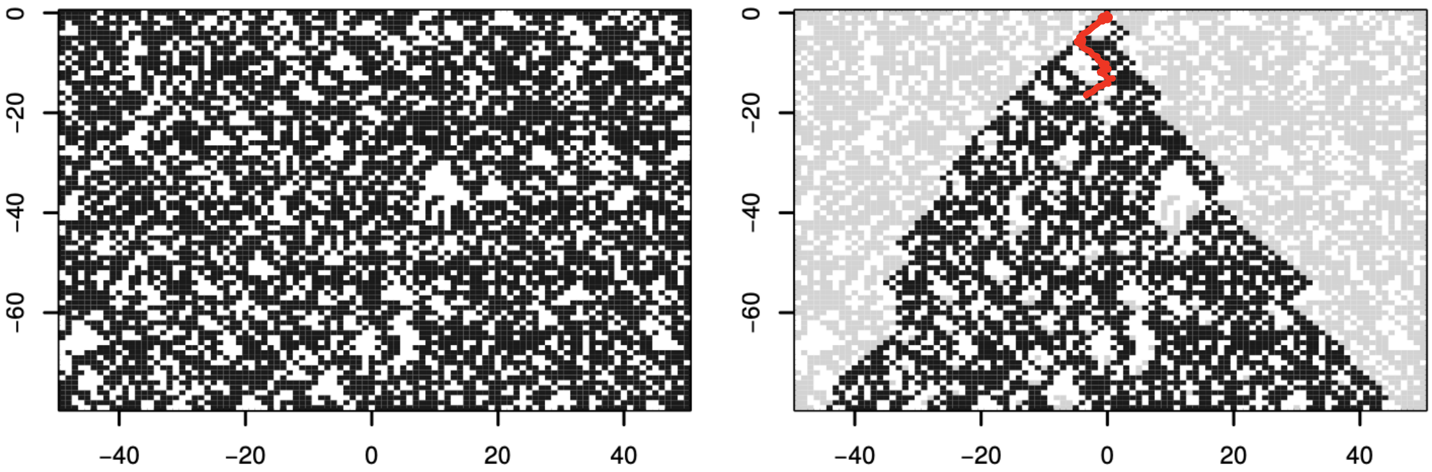


FIGURE 2.2. Left: A simulation of the space-time configuration of the stationary contact process η from (2.3) with $p = 0.68$. Dark sites have $\eta_n(x) = 1$. Right: The same configuration with only those sites highlighted in dark which are potential ancestors of the individual at the origin $(\mathbf{0}, 0)$, i.e. those sites which the walk X with dynamics (2.5) can reach.

Picture taken from `https://arxiv.org/abs/1912.02558` or SIS

- The contact process is also often used as a toy-model for spreading of an infection.

- With this interpretation, we study the spatial location of the carrier of the infection from which it propagated.

Intermezzo

Random Walk in Random Environment

- Static case: - well understood on $\mathbb{Z}$; slowdown-behavior
  - many challenges still remaining in higher dim
    e.g. iid enviroment; does LLN hold?

  Sinai RW

- Dynamic case: - generally, resembles a SRW when the dynamics mixes "fast"

- Both: well understood for certain special models
  e.g. random conductance models

Intermezzo

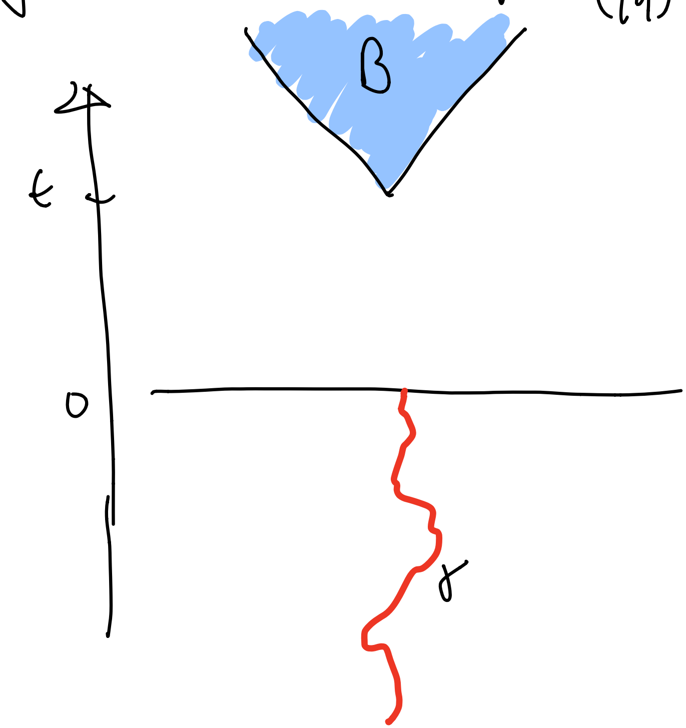
Theorem (B. and Völlering '16)

translation invariant and ergodic

The RW (X_n) satisfies a SLLN if the following mixing condition holds for (η_n):

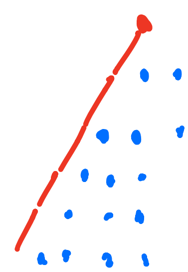
$$\sup_{\gamma} \sup_{B,A} |\mathbb{P}(B|A) - \mathbb{P}(B)| \to 0 \quad \text{as } t \to \infty$$

where A is any event "on γ"



Modelling ancestral lineages

- the discrete-time contact process is not "fast mixing"
- in fact, backwards in time, η_n is not even Markovian
- it is an "non-elliptic" model



Thm (Birkner, Cerny, Depperschmidt and Gantert (2013))

(X_n) on $\mathbb{Z}^d$ satisfies a quenched CLT:

there exist a covariance matrix Σ such that for $\mathbb{P}$-a.e. realization of η_n

$$\frac{X_n}{\sqrt{n}} \Rightarrow W \sim N(0, \Sigma)$$

Work in progress

Thm (B., Birkner, Depperschmidt, Schlüter)

Let $d \geq 3$ and $p > p_c$.

Then $\exists!$ $\varphi \in L_1(\mathbb{P})$, so that for $\mathbb{P}$-a.e. η

$$\lim_{n \to \infty} \sum_{x \in \mathbb{Z}^d} \left| P_\eta^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = x)\, \varphi(\sigma_{(x,n)} \eta) \right| = 0$$

- Here, $P_\eta^{(0,0)}$ is the quenched law and $\mathbb{P}^{(0,0)}$ is the annealed law:

$$\mathbb{P}^{(0,0)}(A) = \int P_\omega^{(0,0)}(A)\, \mathbb{P}(d\omega)$$

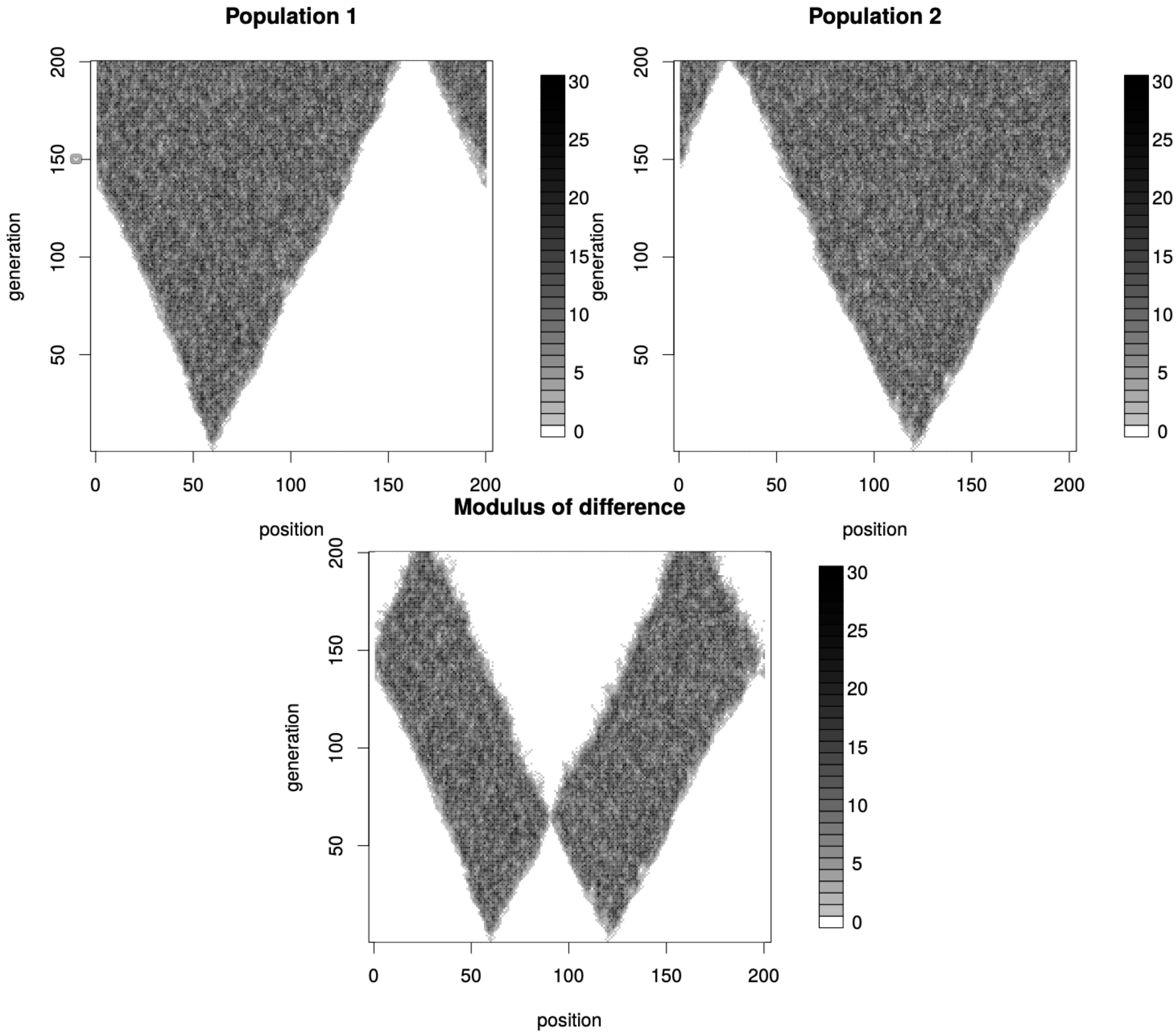
- The theorem is a quenched local CLT.

Extensions to more realistic models

- In 2016, a annealed CLT was proven by Birkner, Černý and Depperschmidt for RWs on a large class of population models:
- E.g. logistic branching random walks:
  - $\eta_n(x)$ = number of individuals at (x,n)
  - Each individual has a Poisson number of offsprings with mean $f(x;\eta_n)/\eta_n(x)$

$$f(x,\zeta) = \zeta(x)\Big(m - \lambda_0\,\zeta(x) - \sum_{z\neq x}\lambda_{xz}\,\zeta(z)\Big)^+$$

  - Each individual jumps according to a SRW
- In a work in progress (jointly T. Schlüter), they also prove a quenched CLT.
- Thus, on large scales, ancestral lineages behave has simple random walks.



Picture taken from `https://arxiv.org/abs/1912.02558`

Proof techics

- Regeneration times:
  - finding suitable random times at which the RW "forgets" its past
- The "environment seen from the walker"-process
- Coupling methods
- Estimates on the behavior of the RW and the population model.