Dynamics and classification
of $C^*$-algebras

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Nuclear C$^*$-algebras

Algebraic and topological regularity

Topological dimension

The Jiang–Su algebra

A conjecture

Some results

Dynamics and dimension
DEFINITION

A $C^*$-algebra is a Banach $*$-algebra $A$ satisfying
\[ \|a^*a\| = \|a\|^2 \]
for all $a \in A$.

(Equivalently, a $C^*$-algebra is a norm-closed, self-adjoint subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.)

We define the cone of positive elements of $A$ by
\[ A_+ := \{ a^*a \mid a \in A \} . \]

A map
\[ \varphi : A \to B \]
is a $*$-homomorphism, if it is linear, multiplicative, and $*$-preserving. $\varphi$ is completely positive, if it is linear, $*$-preserving, and
\[ \varphi^{(n)} : M_n(A) \to M_n(B) \]
is positive for all $n \in \mathbb{N}$. 

W. Winter (University of Nottingham) Dynamics and classification 19.1.2011 3 / 32
EXAMPLES

- $C_0(X)$ for $X$ locally compact
- $C_0(X) \otimes M_r$
- Sums, pullbacks and inductive limits of the above (AH and ASH algebras)
- For $X$ compact and $\alpha : X \to X$ a homeomorphism,

\[ C(X) \rtimes \mathbb{Z} := C^*(C(X), u \mid u \text{ a unitary with } uf(\cdot)u^* = f(\alpha(\cdot))) \]
**THEOREM** (Choi–Effros)
A is nuclear iff there is a system

\[(A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\varphi_\lambda} A)_\Lambda\]

of (finite dimensional) c.p.c. approximations for \(A\) such that

\[\varphi_\lambda \psi_\lambda \to \text{id}_A\]

pointwise.

We will think of such c.p.c. approximations as noncommutative partitions of unity.
CONJECTURE (Elliott)
Separable nuclear $C^*$-algebras are classified by $K$-theoretic data.

$K$-theory is a (computable) homological invariant based on equivalence classes of projections and of unitaries.

But why nuclear $C^*$-algebras?
REMARKS

- Nuclearity is a flexible concept; it can be characterized in many different ways, which make contact with many areas of operator algebras.

- Finite-dimensional approximations seem promising, but c.p. approximations are not a natural framework to study $K$-theoretic data.
**DEFINITION** (Kadison–Kastler)
Let \( A, B \subset B(\mathcal{H}) \) be \( C^* \)-algebras acting on the same Hilbert space. We write \( d(A, B) < \gamma \), if the unit balls of \( A \) and \( B \) are within \( \gamma \) of each other.
There is also a one-sided version (due to Christensen).

**CONJECTURE** (Kadison–Kastler)
If \( A, B \subset B(\mathcal{H}) \) are separable \( C^* \)-algebras and \( d(A, B) < \gamma \) for some small enough \( \gamma \), then \( A \) and \( B \) are (unitarily) isomorphic.
THEOREM (Christensen–Sinclair–Smith–White–W, 2009)
Let $A, B \subset \mathcal{B}(\mathcal{H})$ be C*-algebras, with $A$ separable and nuclear and $d(A, B) < 10^{-6}$.
Then, $A \cong B$.

THEOREM (Christensen–Sinclair–Smith–White–W, 2009)
For $n \in \mathbb{N}$ there is $\gamma > 0$ such that the following holds:
Let $A, B \subset \mathcal{B}(\mathcal{H})$ be C*-algebras, with $A$ separable and $\dim_{\text{nuc}} A \leq n$, and with $A \subset \gamma B$.
Then, there exists an embedding $A \hookrightarrow B$.

(To appear in Acta; announced in PNAS.)

REMARK The first result shows that nuclear C*-algebras are stable under small perturbations; the same holds for $K$-theoretic invariants.
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Dynamics and dimension
Consider the following regularity properties for a $C^*$-algebra $A$.

(A) $A$ is topologically finite-dimensional.

(B) $A$ absorbs a suitable strongly self-absorbing $C^*$-algebra tensorially.

(Γ) $A$ allows comparison of its positive elements in the sense of Murray and von Neumann.

(Δ) The natural order structures on suitable homological invariants of $A$ are complete in the sense that they are sufficiently unperforated.

What do these properties mean?
How are they related?
What can they do for us?
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The Jiang–Su algebra

A conjecture

Some results

Dynamics and dimension
DEFINITION A c.p.c. map \( \varphi : A \to B \) has order zero, if it respects orthogonality, i.e.

\[
(e \perp f \in A_+ \implies \varphi(e) \perp \varphi(f) \in B_+).
\]
DEFINITION (Zacharias–W; Kirchberg–W)
Let $A$ be a $C^*$-algebra, $n \in \mathbb{N}$. We say $A$ has nuclear dimension at most $n$, $\dim_{\text{nuc}} A \leq n$, if the following holds:

For any $\mathcal{F} \subset A$ finite and any $\varepsilon > 0$ there is an approximation

$$A \xrightarrow{\psi} F \xrightarrow{\varphi} A$$

with $F$ finite dimensional, $\psi$ c.p.c., and

$$\varphi \circ \psi = \mathcal{F}, \varepsilon \ id_A,$$

and such that $F$ can be written as

$$F = F^{(0)} \oplus \ldots \oplus F^{(n)}$$

with c.p.c. order zero maps

$$\varphi^{(i)} := \varphi|_{F^{(i)}}.$$

We say $A$ has decomposition rank at most $n$, $\text{dr} A \leq n$, if in addition the map $\sum_i \varphi^{(i)}$ can be chosen to be contractive.
**DEFINITION**  $A$ has locally finite nuclear dimension (or decomposition rank), if for any finite $\mathcal{F} \subset A$ and $\varepsilon > 0$ there is $B \subset A$ such that $\dim_{\text{nuc}} B < \infty$ (or $\text{dr} B < \infty$) and $\mathcal{F} \subset_{\varepsilon} B$.

**REMARKS**

- We have $\dim_{\text{nuc}} A \leq \text{dr} A \leq \dim_{\text{ASH}} A \leq \dim_{\text{AH}} A$.
- Locally finite nuclear dimension implies nuclearity.
- We do not know of any nuclear $C^*$-algebra which does not have locally finite nuclear dimension.
The Jiang–Su algebra \( \mathcal{Z} \)

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Some results

Dynamics and dimension
The Jiang–Su algebra $\mathcal{Z}$

- is the uniquely determined initial object in the category of strongly self-absorbing $\mathbb{C}^*$-algebras (W, 2009).
- is the uniquely determined separable simple unital $\mathbb{C}^*$-algebra which is infinite dimensional, has a unique trace, has finite decomposition rank and is $KK$-equivalent to $\mathbb{C}$ (W, 2008).
- can be written as a stationary inductive limit

$$\lim_{\to}(\mathbb{Z}_2, \mathbb{Z}_3, \alpha)$$

where

$$\mathbb{Z}_2, \mathbb{Z}_3 = \{f \in C([0, 1], M_2 \otimes M_3) \mid f(0) \in M_2 \otimes 1, f(1) \in 1 \otimes M_3\}$$

and $\alpha$ is a trace-collapsing endomorphism of $\mathbb{Z}_2, \mathbb{Z}_3$ (Rørdam–W, 2008).

- can be written as a universal $\mathbb{C}^*$-algebra with countably many generators and relations (Jacelon–W, 2010).
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Some results

Dynamics and dimension
CONJECTURE (Toms–W)
For a nuclear, separable, simple, unital and nonelementary C*-algebra $A$, t.f.a.e.:

(i) $A$ has finite nuclear dimension

(ii) $A$ is $\mathcal{Z}$-stable (i.e., $A \cong A \otimes \mathcal{Z}$)

(iii) $A$ has strict comparison of positive elements
    (i.e., whenever $a, b \in A_+$ satisfy $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$, then $a \preccurlyeq b$).

REMARKS

(a) (ii) $\implies$ (iii) has been shown by Rørdam

(b) (ii) $\implies$ (i) is known in many cases, using classification results

(c) in the finite case, replace ‘nuclear dimension’ by ‘decomposition rank’ in (i)

(d) (iii) holds iff the Cuntz semigroup is almost unperforated; this is related to $K_0$ being weakly unperforated and may be interpreted as order-completeness of a homological invariant.
Some results

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Some results

Dynamics and dimension
THEOREM (W, 2008; Inventiones)
Let $A$ be separable, simple, unital with finite decomposition rank. Then, $A$ is $\mathcal{Z}$-stable.

This is (i) $\implies$ (ii) of (the finite version of) the previous conjecture.

COROLLARY (using results of Lin, Lin–Niu, W)
Separable, simple, unital, monotracial $C^*$-algebras with finite decomposition rank and UCT satisfy the Elliott conjecture.
**DEFINITION** We say a separable, simple, unital, nuclear C*-algebra $A$ is perfect if

- it has strict comparison
- it is almost divisible

(i.e., for any $a \in A_+$ and $k \in \mathbb{N}$ there is $b \in A_+$ with $k \cdot b \preceq a \preceq (k + 1) \cdot b$).
THEOREM  (W, 2010)
Let $A$ be separable, simple, unital, with locally finite nuclear dimension.
If $A$ is perfect, then $A$ is $\mathcal{Z}$-stable.

This is (iii) $\implies$ (ii) of the conjecture above, at least in the case of
locally finite nuclear dimension and almost divisibility.
COROLLARY (using results of Lin, Lin–Niu, Toms, W)
Simple, unital AH algebras with slow dimension growth satisfy the Elliott conjecture.

(This generalizes Elliott–Gong–Li classification of simple unital AH algebras with very slow dimension growth.)
Nuclear C*-algebras

Algebraic and topological regularity

Topological dimension

The Jiang–Su algebra

A conjecture

Some results

Dynamics and dimension
THEOREM (Toms–W, 2008; PNAS)
Let $X$ be compact, metrizable, infinite, with finite covering dimension. Let $\alpha \in \text{Aut}(C(X))$ be induced by a minimal homeomorphism. Then, $C(X) \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable. Moreover, $\dim_{\text{nuc}} (C(X) \rtimes_\alpha \mathbb{Z})$ is finite.
COROLLARY  The class

\[ \mathcal{E} = \{ \mathcal{C}(X) \rtimes_\alpha \mathbb{Z} \mid X \text{ compact, metrizable, infinite, finite dimensional, } \alpha \text{ induced by a uniquely ergodic, minimal homeomorphism} \} \]

is classified by ordered $K$-theory.
In the remainder we will introduce a dynamical version of topological dimension; the aim is twofold:

- To extend the preceding structure and classification results to larger classes of crossed product $C^*$-algebras.
- To pave the road back from crossed product $C^*$-algebras to the underlying dynamical systems.
DEFINITION (Hirshberg–W–Zacharias)
Let $A$ be a unital $C^*$-algebra and $\alpha$ an automorphism of $A$. We say $\alpha$ has Rokhlin dimension at most $n$, if the following holds:

For every finite subset $\mathcal{F} \subset A$, $k \in \mathbb{N}$ and $\epsilon > 0$ there are c.p.c. order zero maps

$$\varphi^{(i)} : \mathbb{C}^k \oplus \mathbb{C}^{k+1} \to A, \ i = 0, \ldots, n,$$

such that

- $\|\alpha(\varphi^{(i)}(e^{(k)}_j)) - \varphi^{(i)}(e^{(k)}_{j+1})\| < \epsilon$ and $\|\alpha(\varphi^{(i)}(e^{(k+1)}_j)) - \varphi^{(i)}(e^{(k+1)}_{j+1})\| < \epsilon$ for all $j$ (cyclic)
- $\|[\varphi^{(i)}(x), a]\| < \epsilon$ for all $a \in \mathcal{F}$, $i = 0, \ldots, n$, and for all normalized $x \in \mathbb{C}^k \oplus \mathbb{C}^{k+1}$
- $\sum_{i=0}^{n} \varphi^{(i)}(1_k \oplus 1_{k+1}) \geq 1_A.$
PROPOSITION (Hirshberg–W–Zacharias, 2010)
Let $A$ be separable and unital, with finite nuclear dimension. If $\alpha \in \text{Aut}(A)$ has finite Rokhlin dimension, then $A \rtimes_{\alpha} \mathbb{Z}$ has finite nuclear dimension.

THEOREM (Hirshberg–W–Zacharias, 2010)
Let $A$ be separable, unital and $\mathcal{Z}$-stable. Then, there is a dense $G_{\delta}$ set $\Gamma \subset \text{Aut}(A)$ such that each $\alpha \in \Gamma$ has finite Rokhlin dimension.

THEOREM (Hirshberg–W–Zacharias, 2010)
Let $A$ be separable, simple and unital, with finite nuclear dimension. Then, there is a dense $G_{\delta}$ set $\Gamma \subset \text{Aut}(A)$ such that for each $\alpha \in \Gamma$, $A \rtimes_{\alpha} \mathbb{Z}$ is simple, $\mathcal{Z}$-stable, with finite nuclear dimension.
PROOF (Idea)

Let $F_l \subset A$, $l \in \mathbb{N}$, be an increasing sequence of finite subsets with dense union.

Define open subsets of $\text{Aut}(A)$

$$V_{k,l} := \{ \alpha \in \text{Aut}(A) \mid \text{the Rokhlin condition holds for } k, F_l \text{ and } 1/l \}.$$ 

Each $\alpha \in \bigcap_{k,l} V_{k,l}$ will have Rokhlin dimension at most $n$.

If we can show that each $V_{k,l}$ is dense in $\text{Aut}(A)$, then the Baire category theorem will imply that $\bigcap_{k,l} V_{k,l}$ is also dense in $\text{Aut}(A)$ (the latter is completely metrizable).

With $\alpha \in \text{Aut}(A)$ given, find $\sigma_{k,l} \in \text{Aut}(\mathbb{Z})$ such that $\alpha \otimes \sigma_{k,l} \in \text{Aut}(A \otimes \mathbb{Z})$ satisfies the Rokhlin condition for $k, F_l \otimes 1_{\mathbb{Z}}$ and $1/l$.

Find isomorphisms $\varrho_m : A \xrightarrow{\cong} A \otimes \mathbb{Z}$ such that

$$V_{k,l} \ni \varrho_m^{-1} \circ (\alpha \otimes \sigma_{k,l}) \circ \varrho_m \xrightarrow{m \to \infty} \alpha \text{ in } \text{Aut}(A).$$
QUESTIONS
Let $X$ be compact, metrizable, infinite, and $\alpha \in \text{Aut}(C(X))$ induced by a minimal homeomorphism.

(i) If $X$ is finite dimensional, does $\alpha$ have finite Rokhlin dimension?
(ii) If $\dim_{\text{nuc}} A \rtimes_{\alpha} \mathbb{Z} < \infty$, does $\alpha$ have finite Rokhlin dimension?

We have confirmed (i) in a number of cases (based on irrational rotations); we hope to have a general result in due course.

The proof would use $C^*$-algebra techniques; the statement only involves the dynamical system, not the crossed product $C^*$-algebra.