An introduction to higher rank graphs and their C^*-algebras, part 2

Toke Meier Carlsen

NTNU, 17 September 2008

Higher rank graphs

Definition (Kumjian and Pask, 2000)

A k-graph is a pair (Λ, d) where Λ is a small category and $d : \Lambda \to \mathbb{N}_0^k$ is a functor such that

$$\forall \lambda \in \Lambda \forall m, n \in \mathbb{N}_0^k \text{ with } m + n = d(\lambda)$$
$$\exists! \mu, \nu \in \Lambda : \mu\nu = \lambda \ \wedge \ d(\mu) = m \ \wedge \ d(\nu) = n.$$

Cuntz-Krieger Λ-families

We say that a k-graph Λ is *row-finite and with no sources* if $0 < \#\Lambda^n(v) < \infty$ for all $v \in \Lambda^0$ and all $n \in \mathbb{N}_0^k$.

Definition (Kumjian and Pask, 2000)

Let Λ be a row-finite k-graph with no sources. A *Cuntz-Krieger Λ-family* in a C^*-algebra A is a family $(s_\lambda)_{\lambda\in\Lambda}$ of partial isometries satisfying:

1. $(s_v)_{v\in\Lambda^0}$ is a family of mutually orthogonal projections,
2. $s_\lambda s_\mu = s_{\lambda\mu}$ for all $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$,
3. $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$,
4. $s_v = \sum_{\lambda\in\Lambda^n(v)} s_\lambda s_\lambda^*$ for all $n \in \mathbb{N}_0^k$ and all $v \in \Lambda^0$.

The C^*-algebra of a row-finite higher rank graph

Theorem (Kumjian and Pask, 2000)

Let Λ be a row-finite k-graph with no sources. Then there exists a C^*-algebra $C^*(\Lambda)$ and a Cuntz-Krieger Λ-family $(s_\lambda)_{\lambda\in\Lambda}$ satisfying:

1. *$C^*(\Lambda)$ is generated by $(s_\lambda)_{\lambda\in\Lambda}$.*
2. *If A is a C^*-algebra which is generated by a Cuntz-Krieger Λ-family $(t_\lambda)_{\lambda\in\Lambda}$, then there exists a unique $*$-homomorphism from $C^*(\Lambda)$ to A mapping s_λ to t_λ for every $\lambda \in \Lambda$.*

The gauge action of the C^*-algebra of a higher rank graph

Theorem (Kumjian and Pask, 2000)

Let Λ be a row-finite k-graph with no sources. Then there exists an action $\gamma : \mathbb{T}^k \to \mathrm{Aut}(C^*(\Lambda))$ (called the gauge action) of $\mathbb{T}^k$ on $C^*(\Lambda)$ such that

$$\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$$

for $z \in \mathbb{T}^k$ and $\lambda \in \Lambda$.

The gauge-invariant uniqueness theorem for the C^*-algebra of a higher rank graph

Theorem (Kumjian and Pask, 2000)

Let Λ be a row-finite k-graph with no sources. If $\{t_\lambda \mid \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family in a C^*-algebra A, $t_v \neq 0$ for every $v \in \Lambda^0$, and there exists an action β of $\mathbb{T}^k$ on A such that $\beta_z(t_\lambda) = z^{d(\lambda)} t_\lambda$ for $z \in \mathbb{T}^k$ and $\lambda \in \Lambda$, then the unique $*$-homomorphism from $C^*(\Lambda)$ to A mapping s_λ to t_λ for all $\lambda \in \Lambda$ is injective.

The Cuntz-Krieger uniqueness theorem for the C^*-algebra of a higher rank graph

Theorem (Kumjian and Pask, 2000)

Let Λ be an aperiodic, row-finite k-graph with no sources. If $\{t_\lambda \mid \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family in a C^*-algebra A such that $t_v \neq 0$ for every $v \in \Lambda^0$, then the unique $*$-homomorphism from $C^*(\Lambda)$ to A mapping s_λ to t_λ for all $\lambda \in \Lambda$ is injective.

Simple C^*-algebras of a higher rank graphs

Theorem (Kumjian and Pask, 2000)

Let Λ be a row-finite k-graph with no sources. Then $C^*(\Lambda)$ is simple if and only if Λ is aperiod and cofinal.

Purely infinite C^*-algebras of a higher rank graphs

Theorem (Kumjian and Pask, 2000)

Let Λ be an aperiodic, row-finite k-graph with no sources. If there for every $v \in \Lambda^0$ are $\lambda, \mu \in \Lambda$ with $d(\mu) \neq 0$ such that $r(\lambda) = v$ and $s(\lambda) = r(\mu) = s(\mu)$, then $C^*(\Lambda)$ is purely infinite in the sense that every hereditary subalgebra contains an infinite projection.

The difference between what we know about graph C^*-algebras and C^*-algebras of higher rank graphs

- The is no construction of a C^*-algebra of a *non-finitely aligned* higher rank graph.
- We have now general formula for the K-theory of the C^*-algebra of a higher rank graph.
- The C^*-algebra of a higher rank graph can be simple without being AF or purely infinite.
- The C^*-algebra of a higher rank graph can have torsion in K_1.

The difference between what we know about graph C^*-algebras and C^*-algebras of higher rank graphs

- The is no construction of a C^*-algebra of a *non-finitely aligned* higher rank graph.
- We have now general formula for the K-theory of the C^*-algebra of a higher rank graph.
- The C^*-algebra of a higher rank graph can be simple without being AF or purely infinite.
- The C^*-algebra of a higher rank graph can have torsion in K_1.

The difference between what we know about graph C^*-algebras and C^*-algebras of higher rank graphs

- The is no construction of a C^*-algebra of a *non-finitely aligned* higher rank graph.
- We have now general formula for the K-theory of the C^*-algebra of a higher rank graph.
- The C^*-algebra of a higher rank graph can be simple without being AF or purely infinite.
- The C^*-algebra of a higher rank graph can have torsion in K_1.

The difference between what we know about graph C^*-algebras and C^*-algebras of higher rank graphs

- The is no construction of a C^*-algebra of a *non-finitely aligned* higher rank graph.
- We have now general formula for the K-theory of the C^*-algebra of a higher rank graph.
- The C^*-algebra of a higher rank graph can be simple without being AF or purely infinite.
- The C^*-algebra of a higher rank graph can have torsion in K_1.

The difference between what we know about graph C^*-algebras and C^*-algebras of higher rank graphs

- The is no construction of a C^*-algebra of a *non-finitely aligned* higher rank graph.
- We have now general formula for the K-theory of the C^*-algebra of a higher rank graph.
- The C^*-algebra of a higher rank graph can be simple without being AF or purely infinite.
- The C^*-algebra of a higher rank graph can have torsion in K_1.

Irrational rotation algebras as C^*-algebras of higher rank graphs

Theorem (Pask, Raeburn, Rørdam and Sims, 2006)

Let $\theta \in]0,1[$ be irrational. There exists a 2-graph Λ and a full projection $p \in C^*(\Lambda)$ such that $pC^*(\Lambda)p$ is a simple $A\mathbb{T}$-algebra with real-rank zero such that $(K_0(pC^*(\Lambda)p), [p]_0)$ is isomorphic to $(\mathbb{Z} + \theta\mathbb{Z}, 1)$ and $K_1(C^*(\Lambda))$ is isomorphic to $\mathbb{Z}^2$. It thus follows from the classification of $A\mathbb{T}$-algebras that $pC^*(\Lambda)p$ is isomorphic to the irrational rotation algebra A_θ.

Irrational rotation algebras as C^*-algebras of higher rank graphs

Theorem (Pask, Raeburn, Rørdam and Sims, 2006)

Let $\theta \in]0,1[$ be irrational. There exists a 2-graph Λ and a full projection $p \in C^*(\Lambda)$ such that $pC^*(\Lambda)p$ is a simple $A\mathbb{T}$-algebra with real-rank zero such that $(K_0(pC^*(\Lambda)p), [p]_0)$ is isomorphic to $(\mathbb{Z} + \theta\mathbb{Z}, 1)$ and $K_1(C^*(\Lambda))$ is isomorphic to $\mathbb{Z}^2$. It thus follows from the classification of $A\mathbb{T}$-algebras that $pC^*(\Lambda)p$ is isomorphic to the irrational rotation algebra A_θ.

Bunce-Deddens algebras

- A supernatural number is a sequence $m = (m_i)_{i\in\mathbb{N}}$ where each $m_i \in \{1, 2, \dots, \infty\}$. We think of m as the formal product $\prod_{i=1}^{\infty} p_i^{m_i}$ where p_i is the ith prime number. We say m is infinite if $\prod_{i=1}^{\infty} p_i^{m_i} = \infty$, or equivalently, if $\sum_{i=1}^{\infty} m_i = \infty$.
- If m is a supernatural number, then $Q(m)$ denotes the subgroup of $\mathbb{Q}$ consisting of the fractions of the form $x(\prod_{i=1}^{\infty} p_i^{-q_i})$ where $x \in \mathbb{Z}$ and $0 \le q_i \le m_i$.
- There exists for each infinite supernatural number m a unique simple unital A$\mathbb{T}$-algebra A with real-rank zero and $(K_0(A), K_1(A)) = (Q(m), \mathbb{Z})$. This algebra is known as the *Bunce-Deddens algebra of type m*.

Bunce-Deddens algebras

- A supernatural number is a sequence $m = (m_i)_{i\in\mathbb{N}}$ where each $m_i \in \{1, 2, \ldots, \infty\}$. We think of m as the formal product $\prod_{i=1}^{\infty} p_i^{m_i}$ where p_i is the ith prime number. We say m is infinite if $\prod_{i=1}^{\infty} p_i^{m_i} = \infty$, or equivalently, if $\sum_{i=1}^{\infty} m_i = \infty$.
- If m is a supernatural number, then $Q(m)$ denotes the subgroup of $\mathbb{Q}$ consisting of the fractions of the form $x(\prod_{i=1}^{\infty} p_i^{-q_i})$ where $x \in \mathbb{Z}$ and $0 \leq q_i \leq m_i$.
- There exists for each infinite supernatural number m a unique simple unital A$\mathbb{T}$-algebra A with real-rank zero and $(K_0(A), K_1(A)) = (Q(m), \mathbb{Z})$. This algebra is known as the *Bunce-Deddens algebra of type m*.

Bunce-Deddens algebras

- A supernatural number is a sequence $m = (m_i)_{i \in \mathbb{N}}$ where each $m_i \in \{1, 2, \ldots, \infty\}$. We think of m as the formal product $\prod_{i=1}^{\infty} p_i^{m_i}$ where p_i is the ith prime number. We say m is infinite if $\prod_{i=1}^{\infty} p_i^{m_i} = \infty$, or equivalently, if $\sum_{i=1}^{\infty} m_i = \infty$.
- If m is a supernatural number, then $Q(m)$ denotes the subgroup of $\mathbb{Q}$ consisting of the fractions of the form $x(\prod_{i=1}^{\infty} p_i^{-q_i})$ where $x \in \mathbb{Z}$ and $0 \leq q_i \leq m_i$.
- There exists for each infinite supernatural number m a unique simple unital A$\mathbb{T}$-algebra A with real-rank zero and $(K_0(A), K_1(A)) = (Q(m), \mathbb{Z})$. This algebra is known as the *Bunce-Deddens algebra of type m*.

Bunce-Deddens algebras

- A supernatural number is a sequence $m = (m_i)_{i\in\mathbb{N}}$ where each $m_i \in \{1, 2, \ldots, \infty\}$. We think of m as the formal product $\prod_{i=1}^{\infty} p_i^{m_i}$ where p_i is the ith prime number. We say m is infinite if $\prod_{i=1}^{\infty} p_i^{m_i} = \infty$, or equivalently, if $\sum_{i=1}^{\infty} m_i = \infty$.
- If m is a supernatural number, then $Q(m)$ denotes the subgroup of $\mathbb{Q}$ consisting of the fractions of the form $x(\prod_{i=1}^{\infty} p_i^{-q_i})$ where $x \in \mathbb{Z}$ and $0 \leq q_i \leq m_i$.
- There exists for each infinite supernatural number m a unique simple unital A$\mathbb{T}$-algebra A with real-rank zero and $(K_0(A), K_1(A)) = (Q(m), \mathbb{Z})$. This algebra is known as the *Bunce-Deddens algebra of type m*.

Bunce-Deddens algebras as C^*-algebras of higher rank graphs

Theorem (Pask, Raeburn, Rørdam and Sims, 2006)

Let m be an infinite supernatural number. There exists a 2-graph Λ and a full projection $p \in C^*(\Lambda)$ such that $pC^*(\Lambda)p$ is a simple $A\mathbb{T}$-algebra with real-rank zero such that $(K_0(pC^*(\Lambda)p), [p]_0)$ is isomorphic to $(Q(m), 1)$ and $K_1(C^*(\Lambda))$ is isomorphic to $\mathbb{Z}$. It thus follows from the classification of $A\mathbb{T}$-algebras that $pC^*(\Lambda)p$ is isomorphic to the Bunce-Deddens algebra of type m.

Bunce-Deddens algebras as C^*-algebras of higher rank graphs

Theorem (Pask, Raeburn, Rørdam and Sims, 2006)

Let m be an infinite supernatural number. There exists a 2-graph Λ and a full projection $p \in C^*(\Lambda)$ such that $pC^*(\Lambda)p$ is a simple $A\mathbb{T}$-algebra with real-rank zero such that $(K_0(pC^*(\Lambda)p), [p]_0)$ is isomorphic to $(Q(m), 1)$ and $K_1(C^*(\Lambda))$ is isomorphic to $\mathbb{Z}$. It thus follows from the classification of $A\mathbb{T}$-algebras that $pC^*(\Lambda)p$ is isomorphic to the Bunce-Deddens algebra of type m.