(Co)compact objects and duality in triangulated categories

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Outline

Compact objects

Cocompact and 0-cocompact objects

Duality and almost split triangles

Setup

- $\mathcal{T}$ is a 'big' triangulated k-category for some commutative ring k.
- For $\mathcal{S} \subset \mathcal{T}$, write

$$\mathcal{S}^{\perp} = \{T \in \mathcal{T} \mid \mathcal{T}(\mathcal{S}[n], T) = 0 \text{ for each } n\}$$

$$^{\perp}\mathcal{S} = \{T \in \mathcal{T} \mid \mathcal{T}(T, \mathcal{S}[n]) = 0 \text{ for each } n\}.$$

Compact objects

Compactness

An object $C \in \mathcal{T}$ is **compact** if we always have

$$\coprod \mathrm{Hom}(C, X_i) \xrightarrow{\cong} \mathrm{Hom}(C, \coprod X_i).$$

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Examples

Let R be a ring.

— R is compact in $\mathrm{D}(\mathrm{Mod}\, R)$:
$\mathrm{Hom}_{\mathrm{D}}(R, -) \cong \mathrm{H}^0(-)$, and this commutes with coproducts.

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— R is compact in $\mathrm{K}(\mathrm{Mod}\, R)$.

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- R is compact in $\mathrm{K}(\operatorname{Mod} R)$.

- An injective resolution λR of R is compact in $\mathrm{K}(\operatorname{Inj} R)$:
  A quasi-isomorphism $R \longrightarrow \lambda R$ induces an isomorphism

  $$\operatorname{Hom}_{\mathrm{K}}(\lambda R, X) \xrightarrow{\cong} \operatorname{Hom}_{\mathrm{K}}(R, X)$$

  for each complex X of injectives. So λR is compact in $\mathrm{K}(\operatorname{Inj} R)$ since R is compact in $\mathrm{K}(\operatorname{Mod} R)$.

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$\mathcal{T}$ is **compactly generated** if there is a set $\mathcal{C} \subset \mathcal{T}$ of compact objects with $\mathcal{C}^{\perp} = 0$.

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Examples

— $\mathrm{D}(\mathrm{Mod}\, R)$ is compactly generated for any ring R.

— $\mathrm{K}(\mathrm{Inj}\, R)$ is compactly generated if R is noetherian.

— $\mathrm{K}(\mathrm{Mod}\, R)$ is typically not generated by compact objects.

Big guns

Theorem (Beligiannis–Reiten)

Suppose $\mathcal{T}$ has coproducts, and let $\mathcal{S} \subset \mathcal{T}$ be a set of compact objects.

Then the inclusion functor $\mathcal{S}^{\perp} \hookrightarrow \mathcal{T}$ admits a left adjoint.

Big guns

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Suppose $\mathcal{T}$ has coproducts, and let $\mathcal{S} \subset \mathcal{T}$ be a set of compact objects.

Then the inclusion functor $\mathcal{S}^{\perp} \hookrightarrow \mathcal{T}$ admits a left adjoint.

Theorem (Neeman)

If $\mathcal{T}$ is compactly generated, then $\mathcal{T}$ satisfies Brown representability.

That is, if a cohomological functor $F\colon \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$ takes coproducts to products, then

$$F \cong \mathrm{Hom}(-, F_R) \text{ for some } F_R \in \mathcal{T}.$$

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Cocompact and 0-cocompact objects

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Non-examples

— If $\mathcal{A}$ is Grothendieck abelian, then the only cocompact object in $\mathrm{D}(\mathcal{A})$ is 0.

— If k is a field, then the only cocompact object in $\mathrm{K}(\mathrm{Mod}\, k)$ is 0.

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An object $C \in \mathcal{T}$ is **0-cocompact** if $\mathrm{Hom}(\mathrm{holim}\, T_i, C) = 0$ for each sequence

$$\cdots \longrightarrow T_2 \longrightarrow T_1 \longrightarrow T_0$$

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Example

Recall: If k is a field, then the only cocompact object in $\mathrm{K}(\mathrm{Mod}\, k)$ is 0.

However, *each* object in $\mathrm{K}(\mathrm{Mod}\, k)$ is 0-cocompact.

Initial motivation

Recall Beligiannis–Reiten: A set of compacts $\mathcal{S} \subset \mathcal{T} \rightsquigarrow$ a left adjoint of $\mathcal{S}^{\perp} \hookrightarrow \mathcal{T}$.

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Theorem

Suppose $\mathcal{T}$ has products, and let $\mathcal{S}$ be a set of 0-cocompact objects in $\mathcal{T}$.

Then the inclusion functor ${}^{\perp}\mathcal{S} \hookrightarrow \mathcal{T}$ admits a right adjoint.

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$\mathcal{T}$ is **0-cocompactly cogenerated** if there is a set $\mathcal{C} \subset \mathcal{T}$ of 0-cocompact objects with ${}^{\perp}\mathcal{C} = 0$.

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Theorem

Suppose $\mathcal{T}$ is 0-cocompactly cogenerated and ‘enhanced’.

If a cohomological functor $G \colon \mathcal{T} \longrightarrow \mathrm{Ab}$ takes products in $\mathcal{T}$ to products in Ab, then

$$G \cong \mathrm{Hom}(G_L, -) \text{ for some } G_L \in \mathcal{T}.$$

Duality and almost split triangles

Duality

Let I be an injective cogenerator of $\operatorname{Mod} k$. Write $(-)^* = \operatorname{Hom}_k(-, I)$.

Typical examples

- k is artinian and I is an injective envelope of the semisimple $k/\operatorname{rad} k$.
- $k = \mathbb{Z}$ and $I = \mathbb{Q}/\mathbb{Z}$.

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A **partial Serre functor** for a subcategory $\mathcal{X} \subset \mathcal{T}$ is a functor $\mathbb{S}\colon \mathcal{X} \longrightarrow \mathcal{T}$ such that

$$\operatorname{Hom}(X, -)^* \cong \operatorname{Hom}(-, \mathbb{S}X).$$

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Theorem

Let $\mathbb{S}\colon \mathcal{X} \longrightarrow \mathcal{T}$ be a partial Serre functor. Then

- $\mathcal{X}$ consists of compact objects;
- $\mathbb{S}(\mathcal{X})$ consists of 0-cocompact objects.

Corollary

If $\mathcal{T}$ is compactly generated, then $\mathcal{T}$ is also 0-cocompactly cogenerated.

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Proof

Let $\mathcal{C}$ be a compact generating set, and let $C \in \mathcal{C}$. By Brown representability,

$$\mathrm{Hom}(C, -)^* \cong \mathrm{Hom}(-, C_R)$$

for some object $C_R \in \mathcal{T}$.

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$\rightsquigarrow$ there is a partial Serre functor $\mathbb{S}\colon \mathcal{C} \longrightarrow \mathcal{T}$ given by $C \longmapsto C_R$

$\rightsquigarrow$ $\mathbb{S}(\mathcal{C})$ is a set of 0-cocompact cogenerators for $\mathcal{T}$. $\square$

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Example

Let R be a ring. Recall that $\mathrm{D}(\mathrm{Mod}\, R)$ has no cocompact objects.

However, this category is cogenerated by 0-cocompact objects.

0-cocompact objects in homotopy categories

Let R be a ring, and write $\nu = (\mathrm{Hom}_R(-, R))^*$

For $M \in \mathrm{C^b}(\mathrm{mod}\, R)$, let $P_1 \xrightarrow{p} P_0 \longrightarrow M \longrightarrow 0$ be a projective presentation. Put

$$\mathbb{S}M = \mathrm{Ker}(\nu(p))[2].$$

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Theorem

This defines a partial Serre functor $\mathbb{S}\colon \mathrm{K}^{\mathrm{b}}(\mathrm{mod}\, R) \longrightarrow \mathrm{K}(\mathrm{Mod}\, R)$.

Almost split triangles

A triangle

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Theorem (Beligiannis, Krause)

Suppose $\mathcal{T}$ satisfies Brown representability. Then each compact object Z with local endomorphism ring appears in an almost split triangle

$$\tau Z \longrightarrow Y \longrightarrow Z \longrightarrow \tau Z[1].$$

An existence result

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By assumption, we can choose τZ such that there is an isomorphism

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Let $\rho\colon E \longrightarrow E/\mathrm{rad}\, E \hookrightarrow I_Z$ be the canonical map. Then the triangle

$$\tau Z \longrightarrow Y \longrightarrow Z \xrightarrow{\phi(\rho)} \tau Z[1]$$

is almost split. □

Theorem

Let $\mathbb{S}\colon \mathcal{X} \longrightarrow \mathcal{T}$ be a partial Serre functor.

If k is noetherian, then we can choose I such that each $Z \in \mathcal{X}$ with local and k-finite endomorphism ring appears in an almost split triangle

$$\mathbb{S}Z[-1] \longrightarrow Y \longrightarrow Z \longrightarrow \mathbb{S}Z.$$

Global approach

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Idea of proof

By assumption we have an isomorphism $\phi\colon\ \mathrm{End}_{\mathcal{T}}(Z)^* \longrightarrow \mathcal{T}(Z, \mathbb{S}Z)$. Let γ be a non-zero linear form on $\mathrm{End}_{\mathcal{T}}(Z)$ which vanishes on $\operatorname{rad}\mathrm{End}_{\mathcal{T}}(Z)$. In the triangle

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the second map is right almost split.

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the second map is right almost split.

Since k is noetherian and I was chosen cleverly, one can show that the triangle is almost split. $\square$

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Proof

Recall the partial Serre functor $\mathbb{S}\colon \mathrm{K}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \longrightarrow \mathrm{K}(\mathrm{Mod}\,\Lambda)$ given by

$$M \longmapsto \mathrm{Ker}(\nu(p))[2],$$

where $P_1 \xrightarrow{p} P_0 \longrightarrow M \longrightarrow 0$ is a projective presentation.

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— $\mathbb{S}$ becomes an auto-equivalence on $\mathrm{K}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$, with quasi-inverse given by

$$M \longmapsto \mathrm{Cok}(\nu^{-}(i))[-2],$$

where $0 \longrightarrow M \longrightarrow I^0 \xrightarrow{i} I^1$ is an injective copresentation. □

Symmetrized duality

Say that **composition from X to Y is non-degenerate** if

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If $\mathbb{S}\colon \mathcal{X} \longrightarrow \mathcal{T}$ is a partial Serre functor, then composition from X to $\mathbb{S}X$ is non-degenerate for each $X \in \mathcal{X}$.

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Proposition

Suppose A and C have local endomorphism rings, and that one of these is artinian.

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If composition from C to A is non-degenerate, then there is an almost split triangle

$$A[-1] \longrightarrow B \longrightarrow C \longrightarrow A.$$

A converse

Say that **composition from X to Y is non-degenerate** if

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If composition from X to Y is non-degenerate, then X is 0-compact and Y is 0-cocompact.

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Theorem

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Corollary

If $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ is an almost split triangle, then X is 0-compact and Z is 0-cocompact.