The Jordan-Hölder property

for Quillen exact categories

NTNU seminar

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Based on

- [BBGH] R-L.Baillargeon, T.Brüstle, M.Gorsky, S.Hassoun, *On the lattice of weakly exact structures,* (arXiv: 2009.10024, 2020).
- [BHT] T.Brüstle, S.Hassoun, A.Tattar, *Intersection, sum and Jordan-Holder property for exact categories* (J. Pure Appl. Algebra 2020).
- [HSW] S.Hassoun, A.Shah, S.-A.Wegner, *Examples and non-examples of integral categories and the admissible intersection property* (Cahiers de Topologie et Géométrie Différentielle Catégoriques 2020).
- [HR] S.Hassoun, S.Roy, *Admissible intersection and sum property*; (arXiv: 1906.03246, 2019)
- [BHLR] T.Brüstle, S.Hassoun, D.Langford, S.Roy, *Reduction of exact structures* (J. Pure Appl. Algebra 224, 2020, no. 4, 106212, 29pp, (2018).

Plan

PLAN

1. History of relative homology
2. Motivation
3. Exact categories
4. Jordan-Hölder exact categories
5. Examples and counter-examples
6. Admissible intersection and sum categories
7. General intersection, sum and radical
8. Artin-Wedderburn exact categories
9. The case of module categories over Nakayama algebras
10. Jordan-Hölder length function

THE ORIGINS OF RELATIVE HOMOLOGY

History of relative homology

1934 *Baer* introduced Ext for abelian groups

1940 *Baer* defines the Baer sum

1954 *Yoneda* proves *the classification theorem*, a one-to-one correspondence between the equivalence classes of the $n-$fold extensions of B by A and the elements of the abelian group $Ext_{\Lambda}{}^{n}(A, B)$

1955 *Buchsbaum* proves the existence of *Ext* for an exact category having enough projectives or enough injectives

1956 *Cartan and Eilenberg* generalize the notion extension groups

1957 *Buchsbaum* defines the extension functor *Ext without* using the projective and the injective objects

1958 *Hochschild* discusses the analogous of the *Ext* but applicable to a module theory that is *relativized* with respect to a given subring of the basic ring of operators

57-58 *Harrison* and *Heller* discuss similar problems, which make it natural to consider the extension functor on a specific exact categories

History of relative homology

The idea of *relative homological algebra for abstract categories* is about the selection of a class of extensions or, equivalently, a class of monomorphisms and epimorphisms.

History of relative homology

1961 *Butler and Horrocks* study relative homological algebras, but only for abelian categories.

They study how the derived functors behave under reduction of the exact structure

History of relative homology

Recent works:

1993 *Auslander and Ø.Solberg* discuss applying relative homological algebras to representation theory

1999 *Dräxler, Reiten, Smalø, Ø.Solberg + Keller* study the correspondence between exact structures and closed additive bifunctors of *Ext*

2005 *Auslander and Ø.Solberg* develope a general theory of *relative* cotilting modules for artin algebras

Motivation

WHY
DO WE WANT TO STUDY THIS SUBJECT ?

Motivation

Nice length function

Jordan-Hölder length improves [BHLR 18'] $\mathcal{E}-$length function

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Intersection and sum of subobjects

Jacobson radical, trace of subcategories, lattice of subobjects,...

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It leads to new characterisations of the important and popular quasi-abelian (functional analysis) and abelian categories

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Applications

The work by [Berktas, Crivei, Kaynarca, Keskin, Tütn̈cü, 21'] which generalize the theorems of uniqueness of uniform decompositions in abelian categories

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Jordan-Hölder property

[Enomoto 19'] $(\mathcal{A}, \mathcal{E})$ satisfies (JHP) if and only if the $\mathcal{E}-$Grenthendieck group is free

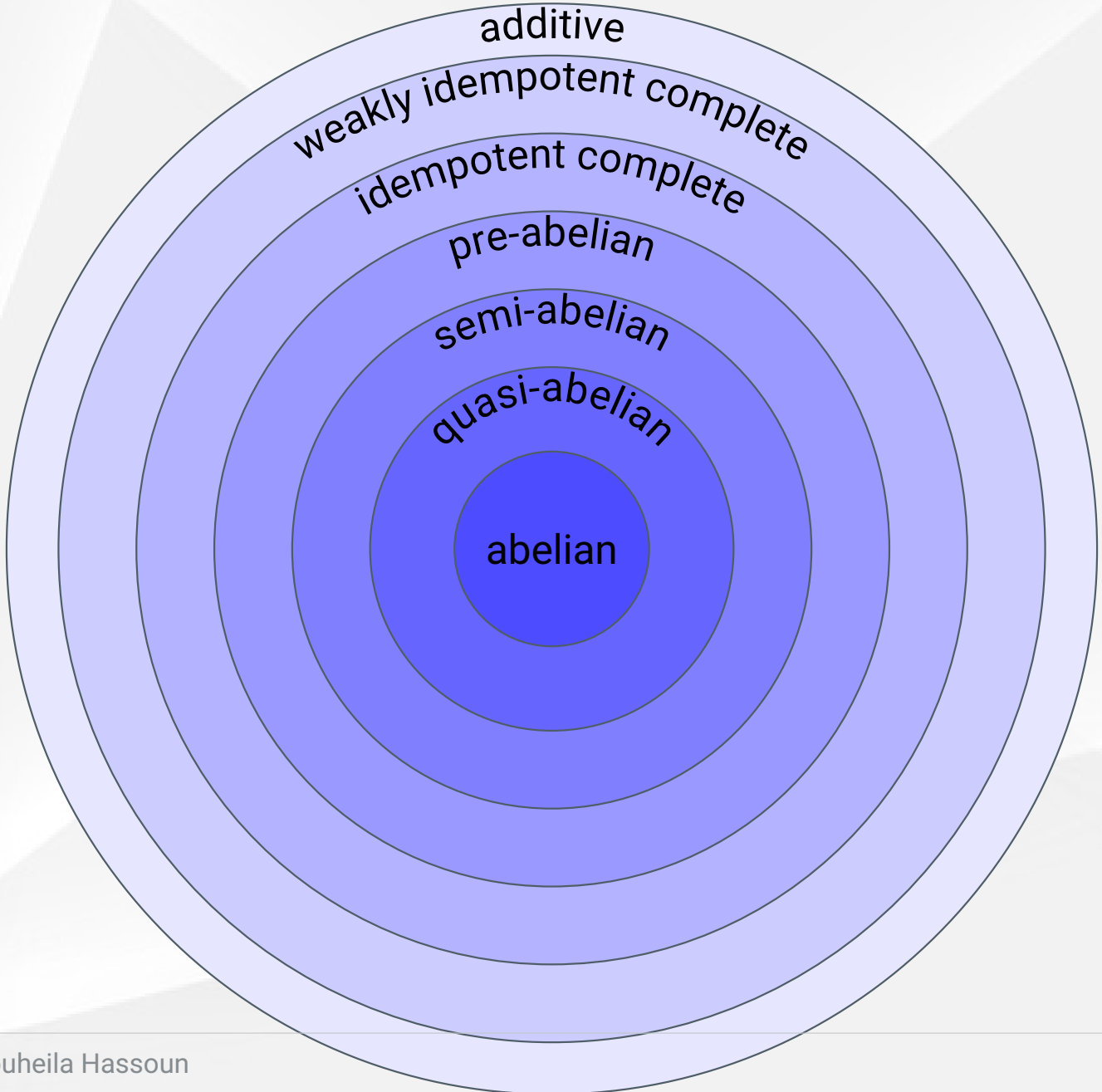
AXIOMATIC DEFINITION

Quillen's exact categories

Definition

A category $\mathcal{A}$ is called *additive* if all its hom-sets are abelian groups and composition of morphisms is bilinear and moreover all finitary products, or finitary coproducts exists.

Additive categories



Quasi-abelian definition

[RW77, Definition]:

A kernel (A, f) is called *semi-stable* if for every push-out square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d, "s_B"] \\ C \arrow[r, "s_C"'] & S \end{tikzcd}$$

the morphism s_C is also a kernel. We define dually a *semi-stable* cokernel. A short exact sequence $A \overset{i}{\rightarrowtail} B \overset{d}{\twoheadrightarrow} C$ is said to be *stable* if i is a semi-stable kernel and d is a semi-stable cokernel. We denote by $_{sta}$ the class of all *stable* short exact sequences.

Definition

An additive category is **quasi-abelian** if it is *pre-abelian* and all kernels and cokernels are *semi-stable*.

Quasi-abelian categories

Example

The category **BAN** of Banach spaces is quasi-abelian.

Quasi-abelian categories

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Examples

Let $k \in \{\mathbb{R}, \mathbb{C}\}$ be fixed. The categories:

NOR of normed spaces;

FRE of Fréchet spaces;

HDLCS of Hausdorff locally convex spaces;

HDTVS of Hausdorff topological vector spaces;

NUC nuclear spaces;

NUCFRE of nuclear Fréchet spaces;

FS of Fréchet-Schwartz spaces;

FH of Fréchet-Hilbert
spaces over k, each furnished with linear and continuous maps as morphisms, are quasi-abelian.

Abelian definition

Definition

An additive category $\mathcal{A}$ is *abelian* if it is *pre-abelian* (has kernels and cokernels) and the induced canonical map

$$\bar{f} : Coim f \to Im f$$

is an isomorphism.

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Examples

The following categories are abelian:

- $\mathbf{Mod_A}$ of $A-$modules or $\mathbf{mod_A}$ of finitely generated ones
- **Ab** of abelian groups
- $\mathbf{Vec_K}$ of $K-$vector spaces
- $\mathbf{Rep(Q)}$ of representation of a quiver Q

Quillen's exact categories

Definition

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(A1) the class of admissible monics (resp. admissible epics) is closed under composition

Quillen's exact categories

Definition

(A2) The push-out of an admissible monic i along an arbitrary morphism a exists and yields an admissible monic s_C:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d, "s_B"] \\ C \arrow[r, tail, "s_C"'] & S \end{tikzcd}$$

(A2)' The pull-back of an admissible epic h along a exists and yields an admissible epic p_B:

$$\begin{tikzcd} P \arrow[r, two heads, "p_B"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\mathrm{PB}"] & B \arrow[d, "a"] \\ A \arrow[r, two heads, "h"] & C. \end{tikzcd}$$

Examples of exact structures

Definition

The short exact sequences isomorphic to

$$A \rightarrowtail A \oplus B \twoheadrightarrow B$$

are called the split ones.

Smallest example

The minimal exact structure formed by all split short exact sequences $\mathcal{E}_{min}$.

Examples of exact structures

[Largest example, Rump 2011]

There exists a unique maximal exact structure $\mathcal{E}_{max}$ for any additive category $\mathcal{A}$.

Example

The maximal exact structure on a quasi-abelian category $\mathcal{A}$ consists of all short exact sequences on $\mathcal{A}$:

$$\mathcal{E}_{max} = \mathcal{E}_{all} = \mathcal{E}_{sta}.$$

The lattice of exact structures

Definition

A poset P is called a *lattice* if for every pair of elements of P there exists a supremum and an infimum. In other words, there exist two binary operations $\vee$ and $\wedge : P \times P \to P$ satisfying the following axioms:

1. $\vee$ is associative and commutative,
2. $\wedge$ is associative and commutative,
3. $\wedge$ and $\vee$ satisfy the following property:

$$m \vee (m \wedge n) = m = m \wedge (m \vee n) \text{ for all } m, n \in P.$$

Theorems [BBGH, 7.34][BHLR, 5.4]:

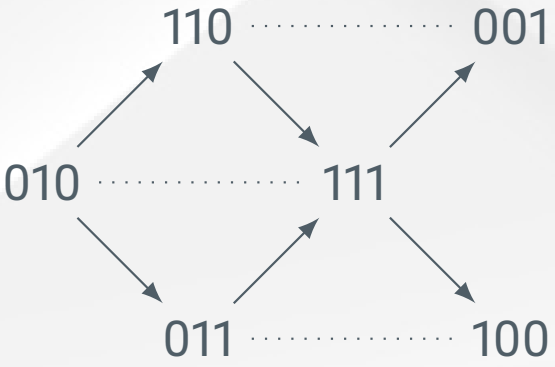
Let $\mathcal{A}$ be an additive category. The map $\Phi : \mathcal{E} \mapsto Ext^1_{\mathcal{E}}(-,-)$ induces a *lattice isomorphism* between $(Ex(\mathcal{A}), \subseteq, \cap, \vee_{Ex})$ and $(Cbf(\mathcal{A}), \leq, \wedge, \vee_{Cbf})$.

Example of $Ex(\mathcal{A})$

Consider the category $\mathcal{A} = repQ$ of representations of $Q = A_3$ quiver

$$Q: \qquad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

The Auslander-Reiten quiver of is as follows:



Example of $Ex(\mathcal{A})$

We consider the following indecomposable non-split exact sequences where the first three are the Auslander-Reiten sequences:

(AR1) $$0 \longrightarrow 110 \longrightarrow 111 \longrightarrow 001 \longrightarrow 0$$

(AR2) $$0 \longrightarrow 011 \longrightarrow 111 \longrightarrow 100 \longrightarrow 0$$

(AR3) $$0 \longrightarrow 010 \longrightarrow 110 \oplus 011 \longrightarrow 111 \longrightarrow 0$$

(4) $$0 \longrightarrow 010 \longrightarrow 110 \longrightarrow 100 \longrightarrow 0$$

(5) $$0 \longrightarrow 010 \longrightarrow 011 \longrightarrow 011 \longrightarrow 0$$

The following list enumerates *all* exact structures $\mathcal{E}$ on $\mathcal{A}$:

- $\mathcal{E}_{min}$ is the class of all split short exact sequences,
- $\mathcal{E}_1 = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in add(AR1)\}$,
- $\mathcal{E}_2 = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in add(AR2)\}$,
- $\mathcal{E}_3 = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in (AR3)\}$,
- $\mathcal{E}_{1,2} = \{X \oplus Y \,| X \in \mathcal{E}_1, Y \in \mathcal{E}_2\}$,
- $\mathcal{E}_{1,3,5} = \{X \oplus Y \oplus Z | X \in \mathcal{E}_1, Y \in \mathcal{E}_3, Z \in add(5)\}$,
- $\mathcal{E}_{2,3,4} = \{X \oplus Y \oplus Z | X \in \mathcal{E}_2, Y \in \mathcal{E}_3, Z \in add(4)\}$,
- $\mathcal{E}_{max}$ is the class of all short exact sequences in .

Example of $Ex(\mathcal{A})$

We have exactly $2^3 = 8$ exact structures and the $Ex(\mathcal{A})$ is a cube:

$$\begin{tikzcd}
 & \mathcal{E}_3 \arrow[rr] \arrow[dl] & & \mathcal{E}_{1,3,5} \arrow[dl] \\
\mathcal{E}_{2,3,4} \arrow[rr] & & \mathcal{E}_{max} & \\
 & \mathcal{E}_{min} \arrow[uu] \arrow[rr] \arrow[dl] & & \mathcal{E}_1 \arrow[uu] \arrow[dl] \\
\mathcal{E}_2 \arrow[uu] \arrow[rr] & & \mathcal{E}_{1,2} \arrow[uu] &
\end{tikzcd}$$

Extriangulated categories

Remark

If $\mathcal{A}$ is skeletally small or has enough projectives or injectives, then $(\mathcal{A}, Ext^1{}_{\mathcal{E}}, \mathbb{I})$ is a Nakaoka-Palu *Extriangulated* category.

JORDAN-HÖLDER PROPERTY

The Jordan-Hölder-Schreier theorem

Theorem

If an $A-$module X admits two composition series

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

and

$$0 = X'_0 \subset X'_1 \subset \cdots \subset X'_{m-1} \subset X'_m = X$$

then they are equivalent: $n = m$ and there exists a permutation σ of $\{0, 1, ..., n-1\}$ such that $X_{i+1}/X_i \cong X'_{\sigma(i)+1}/X'_{\sigma(i)}$.

Examples

The abelian group $\mathbb{Z}_{p^m}$

$$0 = p^m\mathbb{Z}_{p^m} \subset p^{m-1}\mathbb{Z}_{p^m} \subset \cdots \subset p\mathbb{Z}_{p^m} \subset \mathbb{Z}_{p^m}$$

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The group of Klein $V_4 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$< (\bar{0}, \bar{0}) > \subset < (\bar{0}, \bar{1}) > \subset V_4$$
$$< (\bar{0}, \bar{0}) > \subset < (\bar{1}, \bar{0}) > \subset V_4$$
$$< (\bar{0}, \bar{0}) > \subset < (\bar{1}, \bar{1}) > \subset V_4$$

Notations

—> Generalisation	
Abelian categories	Exact categories
subobjects $A \subseteq B$	$\mathcal{E}-$subobjects $A \rightarrowtail B$
simple subobjects $0 \subset S$	$\mathcal{E}-$simple subobject $0 \rightarrowtail S$
Composition series	$\mathcal{E}-$composition series
Jordan-Hölder property Intersection, sum and Jacobson radical Artin-Wedderburn categories	$\mathcal{E}-$Jordan-Hölder property New general intersection, sum and $\mathcal{E}-$radical $\mathcal{E}-$Artin-Wedderburn categories
Lattice of subobject	Poset of subobjects

$\mathcal{E}-$Jordan-Hölder property

Definition

Let $(\mathcal{A}, \mathcal{E})$ be an exact category. A finite $\mathcal{E}-$composition series for an object X of $\mathcal{A}$ is a sequence

$$0 = X_0 \overset{i_0}{\rightarrowtail} X_1 \overset{i_1}{\rightarrowtail} \dots \overset{i_{n-2}}{\rightarrowtail} X_{n-1} \overset{i_{n-1}}{\rightarrowtail} X_n = X$$

where all i are *proper admissible monics* with $\mathcal{E}-$simple cokernel.

$\mathcal{E}-$Jordan-Hölder property

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where all i are *proper admissible monics* with $\mathcal{E}-$simple cokernel. We say an $(\mathcal{A}, \mathcal{E})$ is a *Jordan-Hölder exact category* if any two finite $\mathcal{E}$-composition series of X are equivalent.

Counter-example

Consider $(\mathcal{A}_S, \mathcal{E}_{min})$ with $\mathcal{A}_S = \{v \in Vec_k | dim_k(v) \in S = \mathbb{N} \backslash \{1\}\}$.

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There are two non-equivalent $\mathcal{E}_{min}-$composition series for the object $X = k^6$:

$$0 \longrightarrow K^2 \rightarrowtail K^4 \rightarrowtail K^6$$

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$$0 \longrightarrow K^3 \rightarrowtail K^6.$$

Examples and Counter-examples

Consider $\mathcal{R}ep\mathcal{Q}$ with $\mathcal{Q}$: $1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$

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The *A-R* sequences in $\mathcal{R}ep\mathcal{Q}$ are

- $\mathcal{E}_{min}$ is the class of all split short exact sequences,
- $\mathcal{E}_1 = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in add(AR1)\}$,
- $\mathcal{E}_2 = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in add(AR2)\}$

Examples and Counter-examples

The exact structures accordingly are
$\mathcal{E}_{min}, \mathcal{E}(1), \mathcal{E}(2), \mathcal{E}(3), \mathcal{E}(1,2), \mathcal{E}(1,3), \mathcal{E}(2,3), \mathcal{E}_{max}$

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$(\mathcal{R}ep\mathcal{Q}, \mathcal{E}(1))$ is not Jordan-Hölder since there exists non-equivalent $\mathcal{E}(1)-$composition series $0 \to 010 \to 110 \oplus 011$ and $0 \to 110 \to 110 \oplus 011$

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$(\mathcal{R}ep\mathcal{Q}, \mathcal{E}(2,3))$ is not Jordan-Hölder since there exists non-equivalent $\mathcal{E}(2,3)-$composition series $0 \to 110 \to 111$ and $0 \to 011 \to 111$.

Intersection and sum

OUR APPROACH:
STUDY THE INTERSECTION AND SUM OF SUBOBJECTS

(AI) categories

We call **A.I** the exact categories admitting **A**dmissible **I**ntersections:

[HR, Definition 4.3]:

An exact category $(\mathcal{A}, \mathcal{E})$ is called an ***AI**-category* if $\mathcal{A}$ is pre-abelian satisfying:

(AI) The pull-back (P, p_A, p_B) of two admissible monics a and b exists and yields two admissible monics:

$$\begin{tikzcd} P \arrow[r, tail, "p_B"] \arrow[d, tail, "p_A"'] \arrow[dr, phantom, "\mathrm{PB}"] & B \arrow[d, tail, "b"] \\ A \arrow[r, tail, "a"] & C. \end{tikzcd}$$

[HR, Definition 4.4]:

An exact category $(\mathcal{A}, \mathcal{E})$ is called an ***AS-category*** if:

(*AS*) The morphism u, given by the universal property of the push-out (P, p_B, p_C) is an admissible monic.

$$\begin{tikzcd}
A \arrow[r, tail, "b"] \arrow[d, tail, "c"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d, tail, "p_B"] \arrow[ddr, tail, bend left, "g"] & \\
C \arrow[r, tail, "p_C"] \arrow[drr, tail, bend right, "j"'] & P \arrow[dr, tail, "u"] & \\
& & D.
\end{tikzcd}$$

(AIS) categories

We call **A.I.S-categories** the exact categories admitting **A**dmissible **I**ntersections and **S**ums:

[HR, Definition 4.5]:

An exact category $(\mathcal{A}, \mathcal{E})$ is an ***AIS**-category* if it is an *AI*-category and moreover, the push-out along these pull-backs yields an admissible monic *u*:

$$\begin{tikzcd}
PB \arrow[r, tail, "i"] \arrow[d, tail, "f"] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d, tail, "l"] \arrow[ddr, tail, "g"] & \\
C \arrow[r, tail, "k"] \arrow[drr, tail, "j"] & PO \arrow[dr, tail, "u"] & \\
& & D.
\end{tikzcd}$$

Quasi-abelian categories

[BHT, Theorem 4.12]:

Every AI-category $\mathcal{A}$ is quasi-abelian.

Quasi-abelian categories

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Every AI-category $\mathcal{A}$ is quasi-abelian.

[Brüstle, Hassoun, Shah, Tattar, Wegner]:

A pre-abelian category $\mathcal{A}$ is quasi-abelian if and only if it has admissible intersections.

Abelian categories

[BHT, Theorem 4.22]:

An exact category $(\mathcal{A}, \mathcal{E})$ is an AIS-category if and only if $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{all}$.

Abelian categories

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An exact category $(\mathcal{A}, \mathcal{E})$ is an AIS-category if and only if $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{all}$.

[BHT, Theorem 3.7]

The following conditions are equivalent:

- $\mathcal{A}$ is an abelian category,
- $(\mathcal{A}, \mathcal{E}_{all})$ is an AIS-category,
- $Hom(\mathcal{A}) = Hom^{ad}(\mathcal{A})$,
- $Hom^{ad}(\mathcal{A})$ is closed under composition,
- $Hom^{ad}(\mathcal{A})$ is closed under addition.

Poset of admissible subobjects

Definition

We denote by $\mathcal{P}_X^{\mathcal{E}}$ the set of isomorphism classes of $\mathcal{E}$-subobjects of X. The relation

$$(Y,f) \leq (Z,g) \iff \exists \begin{tikzcd} Y \arrow[r, tail, "\exists h"] \arrow[dr, tail, "f"'] & Z \arrow[d, tail, "g"] \\ & X \end{tikzcd}$$

turns $(\mathcal{P}_X^{\mathcal{E}}, \leq)$ into a poset.

Generalised Intersection

Definition

Consider two $\mathcal{E}-$subobjects (A, f) and (B, g) of X. We denote the set of all common admissible subobjects of A and B as

$$Sub_X(A, B) := \{ \ (Y, h) \in P_X^{\mathcal{E}} \mid Y \in P_A^{\mathcal{E}}, \ Y \in P_B^{\mathcal{E}} \ \}.$$

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[BHT, Definition 5.5]

We define the $\mathcal{E}-$intersection of (A, f) and (B, g) in $P_X^{\mathcal{E}}$ as

$$I_X(A, B) := Max(Sub_X(A, B)).$$

Generalised Sum

Definition

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Generalised Sum

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[BHT, Definition 5.5]

We define the $\mathcal{E}-$sum of A and B in $P_X^{\mathcal{E}}$ as

$$Sum_X(A,B) := Min(Sup_X(A,B)).$$

Examples

Let $(\mathcal{A} = \mathcal{V}_k{}^E, \mathcal{E}_{min})$ be the category of all even dimension k-vector spaces.

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Let $(\mathcal{A} = \mathcal{V}_k{}^E, \mathcal{E}_{min})$ be the category of all even dimension k-vector spaces.

Consider the object $X = k^6 =< v_1, v_2, v_3, v_4, v_5, v_6 >$ and

$$V_1 =< v_1, v_2, v_3, v_4 > \quad \text{and} \quad V_2 =< v_2, v_3, v_4, v_5 > .$$

Examples

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Consider the object $X = k^6 =< v_1, v_2, v_3, v_4, v_5, v_6 >$ and

$$V_1 =< v_1, v_2, v_3, v_4 > \quad \text{and} \quad V_2 =< v_2, v_3, v_4, v_5 > .$$

The abelian intersection $V_1 \cap V_2 =< v_2, v_3, v_4 >$ in $\mod k$

BUT

$$Int_X{}^{\mathcal{E}_{min}}(V_1, V_2) = Gr(2, 3)$$

and

$$Sum_X{}^{\mathcal{E}_{min}}(V1, V2) = \{(X, \mathbb{I})\}.$$

$\mathcal{E}-$Jacobson Radical

[BHT, Definition 6.1]

We define the $\mathcal{E}-$*Jacobson radical* to be the generalised intersection

$$rad_{\mathcal{E}}(X) := I_X\{(Y,f) \in \mathcal{S}_X \mid (Y,f) \in Max(\mathcal{S}_X)\}.$$

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For all $X, Y \in$ and $R \xrightarrow{r} X$.

- For all $(R,r) \in rad(X), rad(Coker(r)) = \{0\}$.
- For all $(Z,g) \in \mathcal{S}_X, Z$ is an $\mathcal{E}$-subobject of some $(R,r) \in rad(X)$ if and only if $pg = 0$ for all $\mathcal{E}-$simple quotients $p : X \twoheadrightarrow S$ of X.

Artin-Wedderburn exact structures

[BHT, Definition 6.4]:

An exact structure $\mathcal{E}$ on $\mathcal{A}$ is called *Artin-Wedderburn* if for any object $X \in \mathcal{A}$ the following properties are equivalent:

(AW1) Every sequence in $\mathcal{E}$ of the form $A \rightarrowtail X \twoheadrightarrow X/A$ splits,

(AW2) X is $\mathcal{E}-$semisimple,

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We say in this case that $(\mathcal{A}, \mathcal{E})$ is an $\mathcal{E}-$*Artin-Wedderburn category*.

Split Artin-Wedderburn structure

The split exact structure $\mathcal{E}_{min}$ is an *Artin-Weddernburn* exact structure:

[BHT, lemma 6.7]:

Any additive category $\mathcal{A}$ is an $\mathcal{E}_{min}$-Artin-Wedderburn category.

Krull-Schmidt categories

Definition

A category is *Idempotent complete* if every idempotent splits.

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[BHT, Theorem 6.8]:

Let $(\mathcal{A}, \mathcal{E})$ be a Krull-Schmidt $\mathcal{E}$-Artin-Wedderburn category. Then $(\mathcal{A}, \mathcal{E})$ is a Jordan-Hölder exact category.

Nakayama Algebras

Definition

An *uniserial* module M is a module over a ring, such that ${\mathcal{P}_M}^{\mathcal{E}_{max}}$ is a totally ordered set or has a unique composition series.

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Example

$\mathbb{Z}_{p^m}$ is *uniserial* in **Ab**:

$$0 = p^m\mathbb{Z}_{p^m} \subset p^{m-1}\mathbb{Z}_{p^m} \subset \cdots \subset p\mathbb{Z}_{p^m} \subset \mathbb{Z}_{p^m}$$

Nakayama Algebras

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A finite-dimensional algebra Λ is called *Nakayama* if every indecomposable right and left projective Λ-module is uniserial.

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Example

The algebra $k[x]/(x^n)$ for k a field and n a positive integer, is a *Nakayama* algebra.

Nakayama Algebras

[BHT, Theorem 6.13]:

Let Λ be a Nakayama algebra, and denote $\mathcal{A} = mod\Lambda$. Then an exact category $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}-$Artin-Wedderburn precisely when it is Jordan-Hölder.

Nakayama Algebras

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Example

Let T_f be a torsion-free class of $mod\Lambda$. Then T_f is an Artin-wedderburn and a Jordan-Hölder exact category.

$\mathcal{E}_{JH}-$length function

[BHT, Definition 7.1]

We define the $\mathcal{E}-$Jordan-Hölder length $l_{\mathcal{E}_{JH}}(X)$ of an object X in as the length of an $\mathcal{E}-$composition series of X. That is $l_{\mathcal{E}_{JH}}(X) = n$ if and only if there exists an $\mathcal{E}-$composition series

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots \qquad \rightarrowtail X_{n-1} \rightarrowtail X_n = X$$

We say in this case that X is $\mathcal{E}-$finite.

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Clearly, isomorphic objects have the same length, and therefore this definition gives rise to a length function $l_{\mathcal{E}} : Obj\mathcal{A} \to \mathbb{N} \cup \{\infty\}$ defined on isomorphism classes.

Examples

For $\mathcal{A} = rep(A_3)$ we recall the following indecomposable non-split exact sequences:

(AR1) $$0 \longrightarrow 110 \longrightarrow 111 \longrightarrow 001 \longrightarrow 0$$

(AR2) $$0 \longrightarrow 011 \longrightarrow 111 \longrightarrow 100 \longrightarrow 0$$

(AR3) $$0 \longrightarrow 010 \longrightarrow 110 \oplus 011 \longrightarrow 111 \longrightarrow 0$$

(4) $$0 \longrightarrow 010 \longrightarrow 110 \longrightarrow 100 \longrightarrow 0$$

(5) $$0 \longrightarrow 010 \longrightarrow 011 \longrightarrow 011 \longrightarrow 0$$

So

$$l_{\mathcal{E}_{min}}(111) = l_{\mathcal{E}_{min}}[0 \to 111] = 1$$

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Properties of $l_{\mathcal{E}_{JH}}$

Let $(\mathcal{A}, \mathcal{E})$ be an exact category:

[BHT, Corollary 7.2]:

Let

$$X \rightarrowtail Z \twoheadrightarrow Y$$

be an admissible short exact sequence of finite length objects. Then

$$l_{\mathcal{E}}(Z) = l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Y).$$

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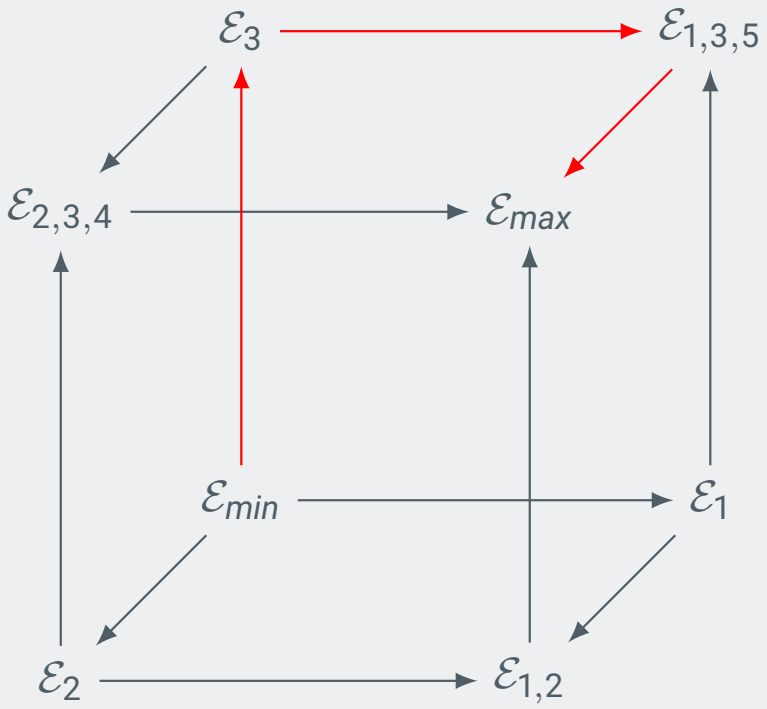
$$l_{\mathcal{E}}(Z) = l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Y).$$

[BHT, Proposition 7.8]:

If $\mathcal{E}$ and $\mathcal{E}'$ are exact structures on $\mathcal{A}$ such that $\mathcal{E}' \subseteq \mathcal{E}$, then $l_{\mathcal{E}'}(X) \leq l_{\mathcal{E}}(X)$ for all objects X in $\mathcal{A}$.

[BHLR, example 8.2]

By taking $Ex(\mathcal{A}) = Ex(Rep(A_3))$:



and notice that the chain of reductions

$$\mathcal{E}_{min} \subseteq \mathcal{E}_{1,3,5} \subseteq \mathcal{E}_{ab}$$

gives us that

$$l_{\mathcal{E}_{min}}(111) = 1 < l_{\mathcal{E}_{1,3,5}}(111) = 2 < l_{\mathcal{E}_{max}}(111) = 3.$$

THANK YOU