Mutation and minimal inclusions of torsion classes

Ongoing joint work with L. Angeleri Hügel, J. Šťovíček and J. Vitória.

0. Overview of talk

Mutation: $\mathcal{M} \xrightarrow{\mu} \mathcal{M}$

$$X_1 \oplus X_2 \oplus \dots \oplus X_n \longmapsto Y_1 \oplus X_2 \oplus \dots \oplus X_n$$

$\downarrow$

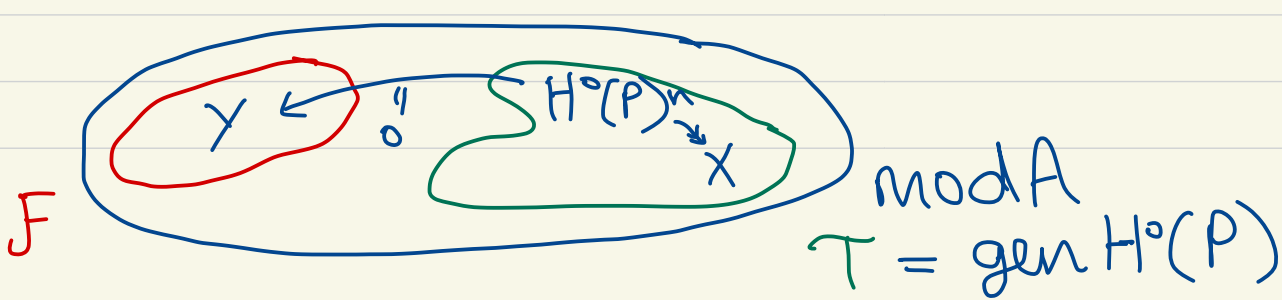
$\text{tors-}A = \{\text{torsion classes in mod}A\}$

$\mathcal{U} \leq \mathcal{T} \iff \mathcal{U} \subseteq \mathcal{T}$

Aim: Hasse(tors-A) — vertices = tors-A
— arrows = minimal inclusions

Known: Hasse(ff-tors-A) $\subseteq$ Hasse(tors-A)

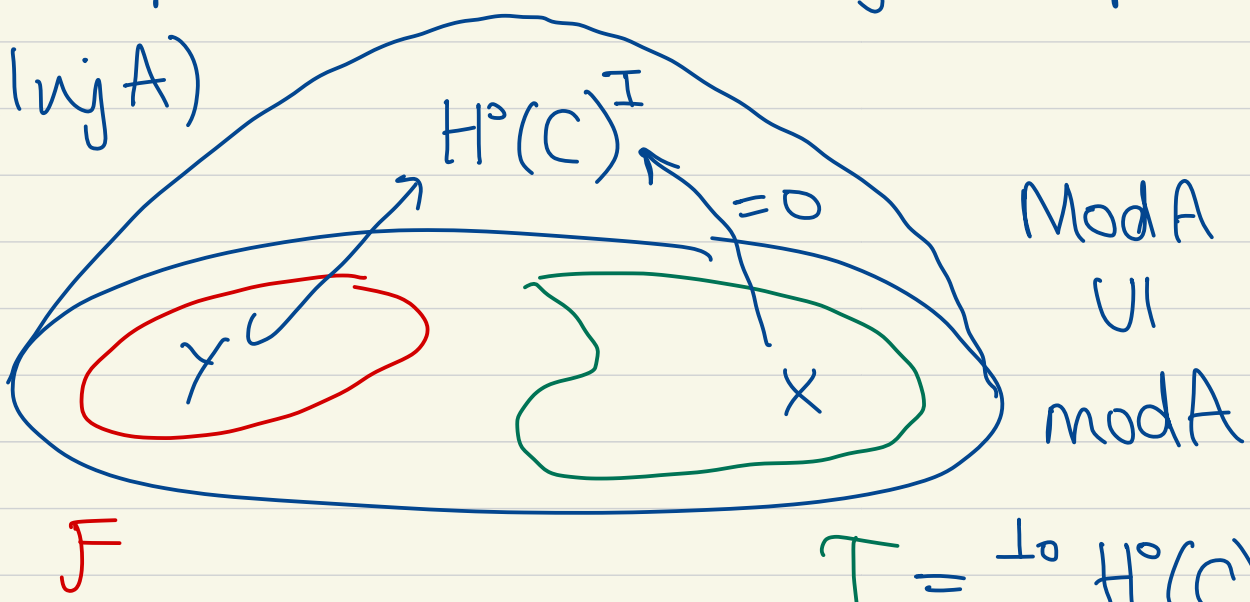
"controlled" by mutation of 2-term silting complexes P in $K^b(\text{proj}A)$



$\mathcal{T} = \operatorname{gen} H^0(P) \supseteq \operatorname{gen} H^0(\mu(P))$

minimal

<u>Today</u>: Hasse(tors-A) "controlled" by mutation of 2-term cosilting complexes C in $K^b(\operatorname{Inj} A)$



$\mathcal{T} = {}^{\perp_0} H^0(C) \supseteq {}^{\perp_0} H^0(\mu(C))$

minimal inclusion

① Introduction

Fix a field $\mathbb{K}$.

- Fix a finite-dimensional $\mathbb{K}$-algebra A

  $\text{tors-}A = \{\text{torsion classes in } \operatorname{mod} A\}$, $\mathcal{U} \leq \mathcal{T} \iff \mathcal{U} \subseteq \mathcal{T}$

Definition: Let $\mathcal{A}$ be an abelian category. A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ is called a torsion pair if

(1) $\operatorname{Hom}_{\mathcal{A}}(T, F) = 0 \quad \forall T \in \mathcal{T}, F \in \mathcal{F}$

(2) $\forall X \in \mathcal{A} \ \exists$ s.e.s. $0 \to \underset{\substack{\uparrow \\ \mathcal{T}}}{t(X)} \to X \to \underset{\substack{\uparrow \\ \mathcal{F}}}{X/t(X)} \to 0$

torsion class $\to \mathcal{T}$

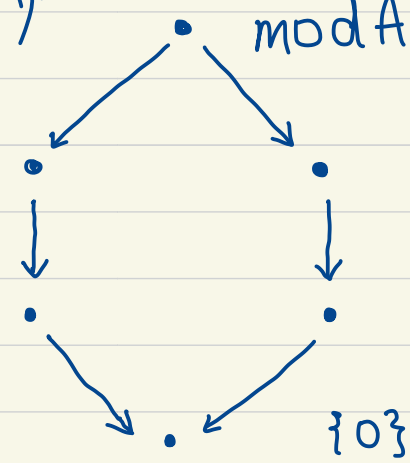
$\mathcal{F}$ $\to$ torsion-free class

- Hasse (tors-A) = quiver with vertices = tors-A
  arrows = minimal inclusions

Examples: ①

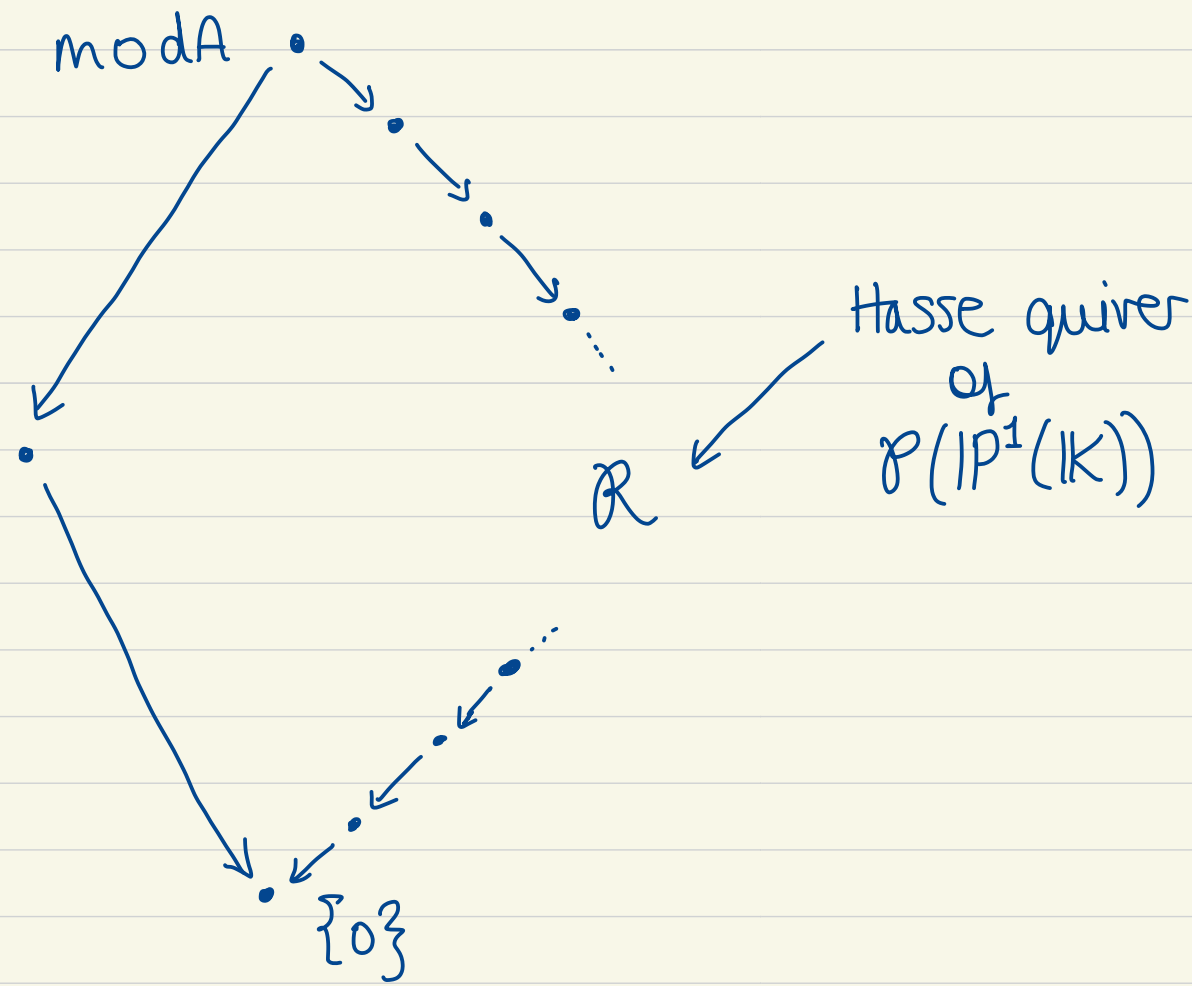
$$A = \mathbb{K}\left(2 \underset{y}{\overset{x}{\leftrightarrows}} 1\right) \Big/ \langle xy, yx \rangle$$

Hasse(tors-A):



② $A = \mathbb{K}\left(1 \rightrightarrows 2\right)$, $\mathbb{K} = \overline{\mathbb{K}}$

Hasse(tors-A):



Definitions: ① $\mathcal{T} \in \text{tors-}A$ is called **functorially finite** if $\forall X \in \text{mod}\,A$ $\exists$ a right and a left $\mathcal{T}$-approximation

$X \in \text{mod}\,A$

$$\begin{tikzcd} \mathcal{T} \ni T_r \arrow[rr, "r"] & & X \\ & T \in \mathcal{T} \arrow[ul, dashed, "\exists"] \arrow[ur, "\forall"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "\ell"] \arrow[dr, "\forall"'] & & T_\ell \in \mathcal{T} \arrow[dl, dashed, "\exists"] \\ & T \in \mathcal{T} & \end{tikzcd}$$

- **ff-tors-A** $= \{$functorially finite torsion classes$\}$

② $P \in K^b(\text{proj}\,A)$ is called a **silting complex** if

$\text{Hom}_{K^b(\text{proj}A)}(P, P[i]) = 0 \;\; \forall i > 0$ and $\text{thick}(P) = K^b(\text{proj}\,A)$

- 2-silt-A $:= \{$basic 2-term silting complexes$\}$

Definition/proposition [Aihara-Iyama 2012]

Let $P = X \oplus Q$ be a silting complex with X indecomp. Consider the triangle

$$Y \longrightarrow Q' \xrightarrow{f} X \longrightarrow Y[1]$$

with f a right $(\text{add}\,Q)$-approximation of X. Then $\mu^-(P) := Y \oplus Q$ is a silting complex called an **irreducible right mutation** of P w.r.t. $\text{add}(Q)$

- $Q(2\text{-silt-}A) :=$ quiver with vertices $=$ 2-silt-A and arrows $=$ irreducible right mutation

<u>Theorem</u> [Adachi-Iyama-Reiten 2014]:

There is an isomorphism of PO sets:

$$\begin{array}{ccc} \text{Hasse}(\text{ff-tors-}A) & \xleftrightarrow{1-1} & Q(2\text{-silt-}A) \\ \text{gen } H^0(P) & \longleftarrow\!\!\shortmid & P \end{array}$$

$$\text{gen } H^0(P) = \{X \in \text{mod}\, A \mid \exists\, H^0(P)^n \twoheadrightarrow X\}$$

$$\mathcal{T} = \text{gen } H^0(P) \supseteq \text{gen } H^0(\mu_Q^-(P))$$

minimal inclusion.

② Cosilting mutation

Definition: A complex $C \in K^b(\mathrm{Inj}\,A)$ is called cosilting if $\mathrm{Hom}_K(C^I, C[i]) = 0$ $\forall i > 0$ $\forall$ sets I and $\mathrm{thick}(\mathrm{Prod}\,C) = K^b(\mathrm{Inj}\,A)$.

$K^b(\mathrm{Inj}\,A) = K$

product of copies of C indexed by I

all $\oplus$ ands of products of copies of C

- $C \sim D$ iff $\mathrm{Prod}\,C = \mathrm{Prod}\,D$
- $2\text{-}\mathrm{Cosilt}\,A := \{2\text{-term cosilting complexes}\}/\sim$

Theorem [Wei-Zhang 2016, Crawley-Boevey 1994]

There is a bijection

$$\begin{array}{ccc} \mathrm{tors}\,A & \xleftrightarrow{1-1} & 2\text{-}\mathrm{Cosilt}\,A \\ \perp_0 H^0(C) & \longleftarrow\!\shortmid & C \end{array}$$

$\perp_0 H^0(C) = \{X \in \mathrm{mod}\,A \mid \mathrm{Hom}_A(X, H^0(C)) = 0\}$

Definition/proposition [ALSV]: Let $C \in K^b(\mathrm{Inj}\,A)$ be a cosilting complex and $\mathcal{E} \subseteq \mathrm{Prod}\,C$ closed under products.

If $\exists E \xrightarrow{f} C$ a minimal right $\mathcal{E}$-approx of C, then

$$G \longrightarrow E \xrightarrow{f} C \longrightarrow G[1]$$

yields a cosilting complex $\mu_{\mathcal{E}}^{-}(C) := G \oplus E$ called the right mutation of C wrt $\mathcal{E}$

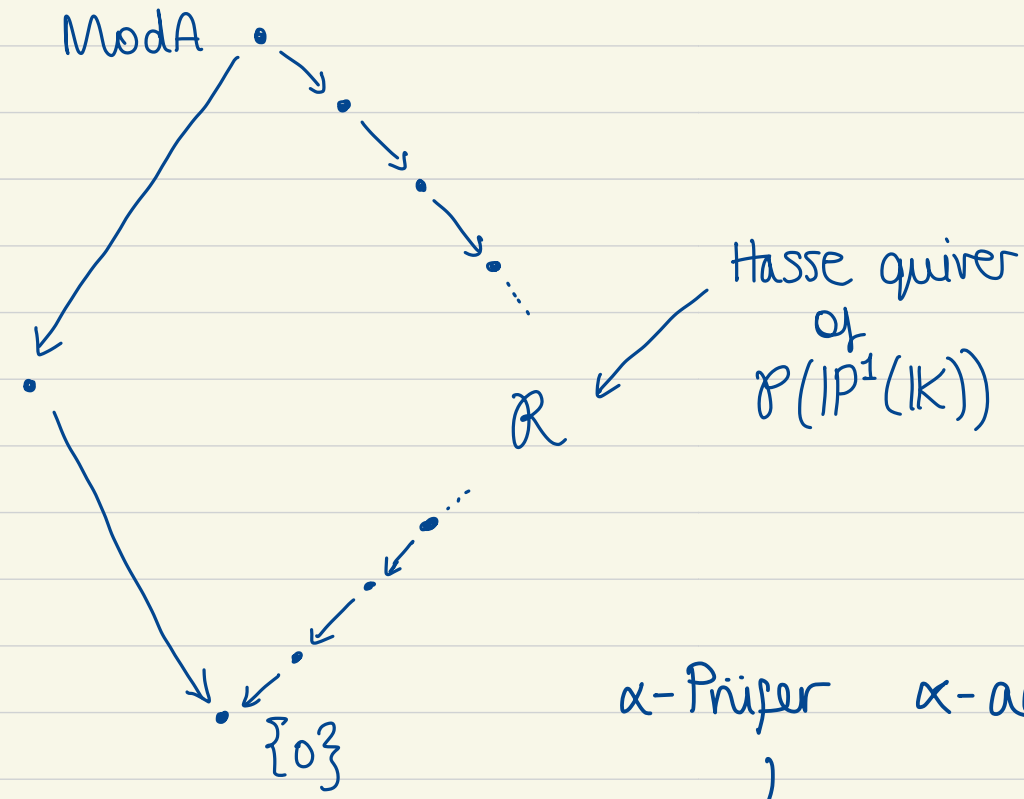
- <u>Question</u>: What is irreducible mutation?

For any indecomposable X such that $C = X \oplus Q$, we can consider $\mathcal{E} := \operatorname{Prod} Q \subseteq \operatorname{Prod} C$

- <u>Problem</u>: Does there exist a minimal right $(\operatorname{Prod} Q)$-approximation of C?

In general, the answer is <u>no</u>, but we will be more precise about which sets $\mathcal{E}$ yield mutations in the next section.

Example: $A = \mathbb{K}(1 \rightrightarrows 2)$, $\mathbb{K} = \overline{\mathbb{K}}$



$$\emptyset \neq \mathcal{S} \in \mathcal{P}(\mathbb{P}^1(\mathbb{K})) \rightsquigarrow C = \underbrace{\prod_{\alpha \in \mathcal{S}} S_\alpha[\infty]}_{\alpha\text{-Prüfer}} \oplus \underbrace{\prod_{\beta \in \mathcal{S}^c} S_\beta[-\infty]}_{\alpha\text{-adic}}$$

$$= S_\gamma[\infty] \oplus Q$$

$$\Rightarrow \mu^-_{\mathrm{Prod}(Q)}(C) \sim Q \oplus S_\gamma[-\infty]$$

$$\mathrm{Ind}(\mathrm{Prod}\, C) = \{S_\alpha[\infty] \mid \alpha \in \mathcal{S}\} \cup \{S_\beta[-\infty] \mid \beta \in \mathcal{S}^c\} \cup \{G\}$$

$$\mathrm{Ind}(\mathrm{Prod}\, \mu^-(C)) = \{S_\alpha[\infty] \mid \alpha \in \mathcal{S} \setminus \{\gamma\}\} \cup \{S_\beta[-\infty] \mid \beta \in \mathcal{S}^c \cup \{\gamma\}\} \cup \{G\}$$

③ Irreducible cosilting mutation

Let $D(A) := D(\mathrm{Mod}\,A)$ unbounded derived cat. of $\mathrm{Mod}\,A$

- Silting and cosilting complexes

$\rightsquigarrow$ t-structure in $D(A)$
(i.e. torsion pair $\tau = (\mathcal{U}, \mathcal{V})$ in $D(A)$ with $\mathcal{U}[1] \subseteq \mathcal{U}$)

$\rightsquigarrow$ Heart $\mathcal{H}_\tau := \mathcal{V} \cap \mathcal{U}[-1]$ is abelian and $\exists$ functor $H^0_\tau : D(A) \longrightarrow \mathcal{H}_\tau$ cohomological.

Examples: ① $\mathrm{Mod}\,A \subseteq D(A)$ in degree zero

$H^0 : D(A) \longrightarrow \mathrm{Mod}\,A$ zeroth cohomology

② P silting complex in $K^b(\mathrm{proj}\,A) \subseteq D(A)$

$\mathcal{H}_P := P^{\perp_{\neq 0}} = \{X \in D(A) \mid \mathrm{Hom}_{D(A)}(P, X[i]) = 0 \ \forall i \neq 0\}$

Denote $H^0_P : D(A) \longrightarrow \mathcal{H}_P$.

Note: $\mathcal{H}_P \cong \mathrm{Mod}\text{-}B$, $B = \mathrm{End}_{D(A)}(P)$ f.d. alg

$H^0_P : \mathrm{Add}\,P \xrightarrow{\sim} \mathrm{Add}\,H^0_P(P) = \mathrm{Proj}\,\mathcal{H}_P$.

③ C cosilting in $K^b(\mathrm{Inj}\,A) \subset D(A)$

$\mathcal{H}_C := {}^{\perp_{\neq 0}} C = \{X \in D(A) \mid \mathrm{Hom}_{D(A)}(X, C[i]) = 0 \ \forall i \neq 0\}$

Denote $H^0_C : D(A) \longrightarrow \mathcal{H}_C$.

<u>Note</u>: $H^0_C : \mathrm{Prod}\,C \xrightarrow{\sim} \mathrm{Prod}\,H^0_C(C) = \mathrm{Inj}\,\mathcal{H}_C$

$\mathcal{H}_C$ Grothendieck abelian.

④ Let $\mathcal{H}_\tau$ be the heart of a t-structure, $t = (\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{H}_\tau$

$\mathcal{H}_t := \{X \in D(A) \mid H^0_\tau(X) \in \mathcal{F},\ H^1_\tau(X) \in \mathcal{T},\ H^i_\tau(X) = 0 \ \forall i \neq 1, 0\}$

HRS-tilt of $\mathcal{H}_\tau$ wrt t.

Happel-Reiten-Smalø

Theorem [Koenig–Yang 2013]: $P = Q \oplus X \in K^b(\mathrm{proj}\,A)$ silting, X indecomp.

$\rightsquigarrow$ $H_P^0(X) \in \mathrm{Proj}\,\mathcal{H}_P$ indecomp.

Let $S = \mathrm{top}\, H_P^0(X)$ and let

$$t := ({}^{\perp_0}(S^{\perp_0}), S^{\perp_0}) = (\mathrm{Filt}\,S, S^{\perp_0}) \text{ in } \mathcal{H}_P$$

Then

$$\mathcal{H}_{\mu_Q^-(P)} = \mathcal{H}_t$$

silting $\rightarrow$ $\qquad$ $\leftarrow$ HRS-tilt

Theorem [ALSV]: Let $C, D \in K^b(\mathrm{Inj}\,A)$ be cosilting complexes. TFAE:

① $D \sim \mu_{\mathcal{E}}^-(C)$ wrt some $\mathcal{E} \subseteq \mathrm{Prod}\,C$

② $\mathcal{H}_D = \mathcal{H}_t$ where $t = (\mathcal{T}, \mathcal{F})$ is some hereditary torsion pair of finite type in $\mathcal{H}_C$.

$\mathcal{T}$ closed under $\hookrightarrow$

$\mathcal{F}$ closed under $\varinjlim$

<u>Example</u>: Let $C \in K^b(\mathrm{Inj}A)$ be a cosilting complex and suppose $S \in \mathcal{H}_C$ is a finitely presented simple object. Then $t := (\mathrm{Filt}\,S, S^{\perp_0})$ is a hereditary torsion pair of finite type in $\mathcal{H}_C \rightsquigarrow \exists$ mutation $\mu_S^-(C)$.

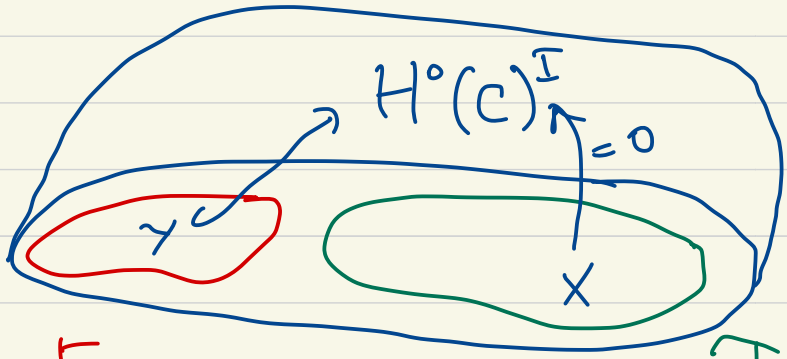
<u>Definition</u> (today): $C \in K^b(\mathrm{Inj}A)$, S fp simple in $\mathcal{H}_C$. We will call $\mu_S^-(C)$ an irreducible right mutation of C.

- $Q(2\text{-}\mathrm{Cosilt}A) :=$ quiver $\begin{cases} \text{vertices: } 2\text{-}\mathrm{Cosilt}A \\ \text{arrows: irreducible right mutation} \end{cases}$

<u>Theorem</u> [ALSV]: $\exists$ isomorphism of posets:

$$\mathrm{Hasse}(\mathrm{tors}\text{-}A) \xleftrightarrow{\cong} Q(2\text{-}\mathrm{Cosilt}A)$$

$$^{\perp_0}H^0(C) \longleftarrow\!\shortmid\; C$$



$\mathrm{Mod}\, A$
$\cup\!\!\shortmid$
$\mathrm{mod}\, A$

$\mathcal{F} = \mathrm{Cogen}\, H^0(C) \cap \mathrm{mod}\, A$

$\mathcal{T} = {}^{\perp_0} H^0(C)$

$$\mathcal{T} = {}^{\perp_0} H^0(C) \supseteq {}^{\perp_0} H^0(\mu_S^-(C))$$

minimal inclusion.