The wall and chamber structure of an algebra: a geometric approach to τ-tilting theory

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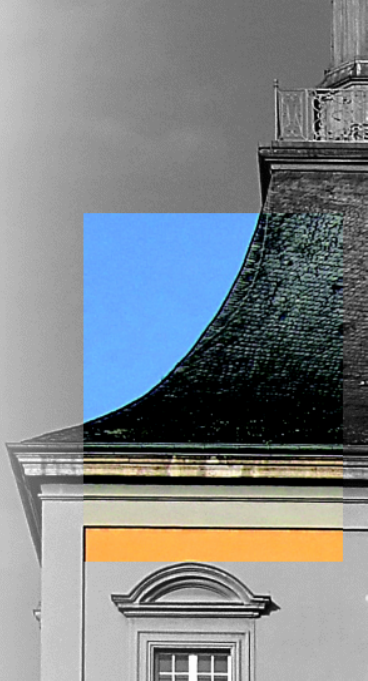
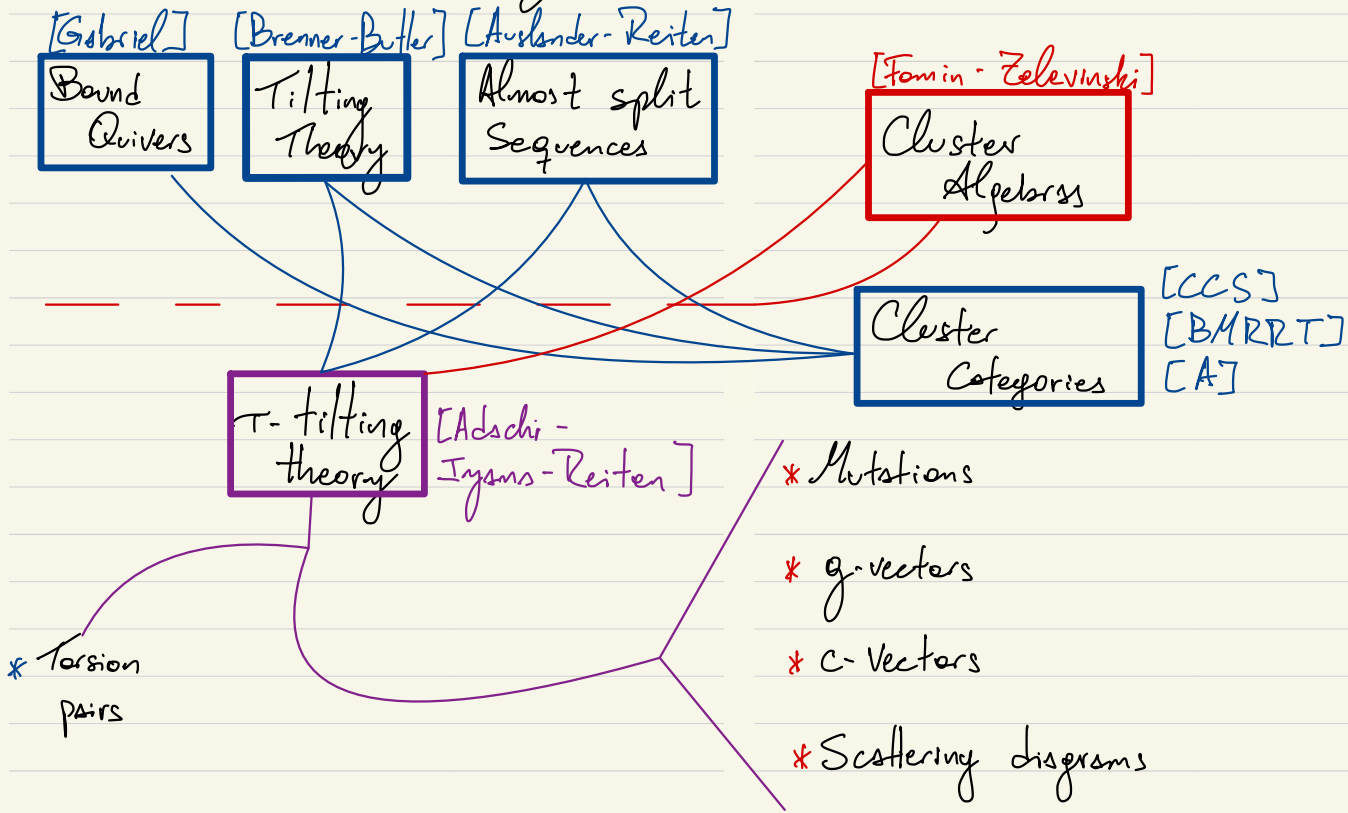


Plan of the talk:

* Introduction
  - A bit of History
  - Setting and notation
* Prelude: Torsion classes
* τ-tilting theory
* g-vectors and τ-tilting theory
* Wall and chamber structures
* c-vectors and (semi)bricks
* The first τ-Brauer-Thrall conjecture

A bit of History



Setting

* A is a basic fd algebra over k
* $\operatorname{mod} A$ fp right A-modules
* τ is the Auslander-Reiten translation
* $K_0(A)$ Grothendieck group $\operatorname{mod} A$

Notation

* $A = \bigoplus_{i=1}^{n} P(i)$
* $|M| = \#$ iso-classes ind summands of M
* $|A| = \operatorname{rk}(K_0(A)) = n$
* $\operatorname{Fac} M = \{ X \in \operatorname{mod} A : M^t \to X \to 0 \quad t \in \mathbb{N} \}$

Prelude: Torsion Pairs

Def: [Dickson]

A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\operatorname{mod} A$ is a torsion pair if

(a) $\operatorname{Hom}_A(X, Y) = 0 \quad \forall X \in \mathcal{T}, Y \in \mathcal{F}$

(b) For every $M \in \operatorname{mod} A$

$$\begin{tikzcd} 0 \arrow[r] & tM \arrow[r] & M \arrow[r] & fM \arrow[r] & 0 \\ & \mathcal{T} \arrow[u, phantom, "\in"] & & \mathcal{F} \arrow[u, phantom, "\in"] & \end{tikzcd}$$

In this case we that $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion free class.

Theorem [Dickson]

A subcategory $\mathcal{T}$ of $\operatorname{mod} A$ is a torsion class if and only if $\mathcal{T}$ is closed under quotients and extensions.

Proposition

Let $M \in \operatorname{mod} A$. $\operatorname{Filt}(\operatorname{Fac}(M)) = T(M)$

$$T(M) = \{ X \in \operatorname{mod} A \mid \exists\, 0 \subset X_1 \subset X_2 \subset \dots \subset X_s = X \quad X_i / X_{i-1} \in \operatorname{Fac} M \}$$

is a torsion class.

Remark: $\operatorname{Fac}(\operatorname{Filt}(M))$ is not a torsion class in general.

τ-tilting theory

Def: [Adachi-Iyama-Reiten]

Let $M, P \in \operatorname{mod} A$ with P projective.

- M is τ-rigid if $\operatorname{Hom}_A(M, \tau M) = 0$
- A pair (M, P) is τ-rigid if M is τ-rigid and $\operatorname{Hom}_A(P, M) = 0$
- A τ-rigid pair (M, P) is τ-tilting if $|M| + |P| = n$.

Theorem [Auslander-Smalø]

Let $M \in \operatorname{mod} A$. Then $\operatorname{Fac} M$ is a torsion class if and only if M is τ-rigid.

Rmk: $\operatorname{Fac} M$ is functorially finite.

Theorem [Auslander-Smalø][Adachi-Iyama-Reiten]

There is a well-defined map

$$\begin{aligned} \Phi : \tau\text{-rig}(A) &\longrightarrow \operatorname{fftors}(A) \\ (M, P) &\longmapsto \operatorname{Fac} M. \end{aligned}$$

Moreover Φ is bijective if restricted to the set $\tau\text{-tilt}(A) \subset \tau\text{-rig}(A)$.

• **Notation:** $\mathcal{M}^{\perp} = \{X : \mathrm{Hom}_A(\mathcal{M}, X) = 0\}$ $\qquad {}^{\perp}\mathcal{M}$ dual

Let $(\mathcal{M}, \mathcal{P})$ be τ-rigid.

$\to (\mathrm{Fac}\,\mathcal{M}, \mathcal{M}^{\perp})$ Bongartz cocompletion

$\to ({}^{\perp}\tau\mathcal{M}) \cap (\mathcal{P}^{\perp}), \mathcal{F}^{*})$ Bongartz completion

Theorem: [Jasso] (τ-tilting reduction)

Let $(\mathcal{M}, \mathcal{P})$ be a τ-rigid pair. Then the perpendicular category of $(\mathcal{M}, \mathcal{P})$

$$\mathcal{M}^{\perp} \cap {}^{\perp}\tau\mathcal{M} \cap \mathcal{P}^{\perp}$$

is equivalent to the module category $\mathrm{mod}\, B_{(\mathcal{M},\mathcal{P})}$ of an algebra $B_{(\mathcal{M},\mathcal{P})}$.

Definition: [Demonet - Iyama - Jasso]

An algebra A is τ-tilting finite if there are finitely many τ-tilting pairs in $\mathrm{mod}\, A$.

Theorem [Demonet - Iyama - Jasso]

An algebra A is τ-tilting finite if and only if every torsion class in $\mathrm{mod}\, A$ is functorially finite.

g-vectors and τ-tilting theory

Def:
Let $M \in \operatorname{mod} A$ and let $P_1 \to P_0 \to M \to 0$ its minimal projective presentation.

$$P_0 = \bigoplus_{i=1}^{n} P(i)^{a_i} \qquad P_1 = \bigoplus_{i=1}^{n} P(i)^{b_i}$$

$$g^M = (a_1 - b_1, a_2 - b_2, \dots, a_n - b_n) \in \mathbb{Z}^n$$

Rk: g-vectors are elements of $K_0(K^{[a,-]}(\operatorname{proj} A))$ which is a hereditary extriangulated category

Theorem: [Auslander-Reiten][Dehy-Keller][AIR][BIJ]

If $M, M' \in \operatorname{mod} A$ are τ-rigid such that $g^M = g^{M'}$ then $M \cong M'$.

Let (M,P) τ-rigid $\quad M = \bigoplus_{i=1}^{\ell} M_i \quad P = \bigoplus_{j=\ell+1}^{t} P_j$

Theorem: [Adachi-Iyama-Reiten][Demonet-Iyama-Jasso]
Let (M,P) be a τ-tilting pair. Then the set

$$\{ g^{M_1}, \dots, g^{M_\ell}, -g^{P_{\ell+1}}, \dots, -g^{P_n} \}$$

is a basis of $\mathbb{Z}^n$

Def: Then the cone $\mathcal{C}_{(M,P)}$ of (M,P) is

$$\mathcal{C}_{(M,P)} = \{ \sum \alpha_i g^{M_i} - \sum \alpha_j g^{P_j} : \alpha_i, \alpha_j > 0 \}.$$

Theorem [Demonet-Iyama-Jasso]
An algebra A is τ-tilting finite iff

$$\bigcup_{\substack{(M,P) \\ \tau\text{-rigid}}} \mathcal{C}_{(M,P)} = \mathbb{R}^n$$

The Auslander-Reiten form

Let $K_0(A)$ the Grothendieck group of A.

$$K_0(A) \xrightarrow{\sim} \mathbb{Z}^n$$
$$[S(i)] \longmapsto e_i$$

$$M \in \operatorname{mod} A \rightsquigarrow [M] \in \mathbb{Z}^n$$

Theorem: [Auslander-Reiten]

Let $M, N \in \operatorname{mod} A$. Then

$$\langle g^M, [N] \rangle = \dim_k(\operatorname{Hom}_A(M,N)) - \dim_k(\operatorname{Hom}_A(N, \tau M))$$

Rmk:

$$\langle -,- \rangle : K_0(K^{[0,1]}(\operatorname{proj} A)) \times K_0(A) \longrightarrow \mathbb{Z}$$

Prop: [Brüstle-Smith-T.] [Yurikusa]

Let (M,P) be τ-rigid and let $v \in \mathcal{C}_{(M,P)}$. Then

$$\operatorname{Fac} M = \{ X \in \operatorname{mod} A : \langle v, Y \rangle \geq 0 \ \forall\ X \to Y \to 0 \}$$

$$^{\perp}\tau M \cap P^{\perp} = \{ X \in \operatorname{mod} A : \langle v, Y \rangle \geq 0 \ \forall\ X \to Y \to 0 \}$$

Proof:

$$\langle g^M - g^P, [X] \rangle = \dim_k(\operatorname{Hom}_A(M,X)) - \dim_k(\operatorname{Hom}_A(X, \tau M)) - \dim_k(\operatorname{Hom}_A(P,X))$$

Wall and chamber structure

Def [King]
Let $M \in \operatorname{mod} A$ and $v \in \mathbb{R}^n$. We say that X is *v-semistable* if

$$\langle v, [X] \rangle = 0$$
$$\langle v, [L] \rangle \leq 0 \quad \forall \; L \hookrightarrow X$$
$$\langle v, [N] \rangle \geq 0 \quad \forall \; X \twoheadrightarrow N$$

Theorem: [Brüstle-Smith-T.] [Yurikusa]
Let (M,P) be τ-rigid and $v \in \mathcal{C}_{(M,P)}$. A module X is v-semistable iff $X \in M^{\perp} \cap {}^{\perp}\tau M \cap P^{\perp}$. In particular $\operatorname{mod}_v^{ss} A$ is equivalent to $\operatorname{mod} B_{(M,P)}$.

Def:
Let $X \in \operatorname{mod} A$. The *stability space* $\mathcal{D}(X)$ of X is

$$\mathcal{D}(X) = \{ v \in \mathbb{R}^n : X \text{ is } v\text{-semistable} \}$$

A *chamber* is an open connected component of

$$\mathbb{R}^n \setminus \bigcup_{0 \neq X \in \operatorname{mod} A} \mathcal{D}(X)$$

A *Wall* is $\mathcal{D}(X)$ of codimension 1.

Theorem [Brüstle-Smith-T.] [Asai]
Let (M,P) be a τ-tilting pair. Then $\mathcal{C}_{(M,P)}$ is a chamber. Moreover every chamber arises this way.

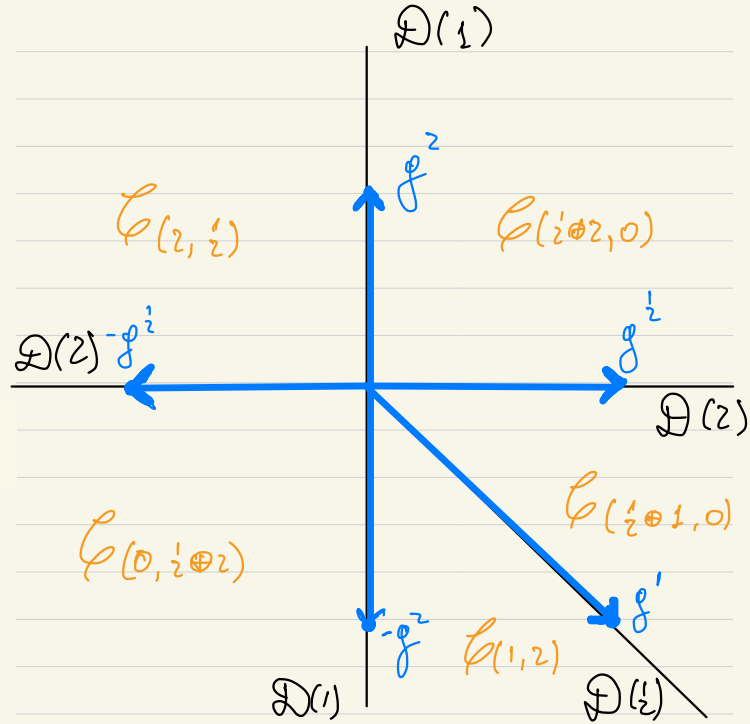
$$1 \longrightarrow 2$$

$$\begin{tikzcd} & \frac{1}{2} \arrow[dr] & \\ 2 \arrow[ur] \arrow[rr, dashed] & & 1 \end{tikzcd}$$

$$\mathcal{D}(1) = \{ (0, y) : \quad y \in \mathbb{R} \}$$

$$\mathcal{D}(2) = \{ (x, 0) : \quad x \in \mathbb{R} \}$$

$$\mathcal{D}(\tfrac{1}{2}) = \{ (x, -x) : \quad x \geq 0 \}$$



C-vectors and bricks

Def: [Fu]
Let (M,P) be τ-tilting.

$$G_{(M,P)} = \left(g^{M_1} \,|\, g^{M_2} \,|\, \dots \,|\, g^{P_n} \right)$$

$$C_{(M,P)} = \left(G_{(M,P)}^{T} \right)^{-1} = \left(c_1 \,|\, c_2 \,|\, \dots \,|\, c_n \right)$$

The set $\{c_1, \dots, c_n\}$ are the c-vectors associated to (M,P).

Corollary
Every wall surrounding $\mathcal{C}_{(M,P)}$ is perpendicular to a c-vector.

Def:
A module B is called a brick if $\mathrm{End}_A(B)$ is a division ring.

Theorem: [T.]

Let A be an algebra over $k = \bar{k}$ and (M,P) be a τ-tilting pair. Then there is a set
$$\{B_1, \dots, B_n\}$$
of bricks such that $c_i = (-1)^{\varepsilon_i}[B_i]$, with $\varepsilon_i \in \{0,1\}$. Moreover $\varepsilon_i = 0$ iff $B_i \in \mathrm{Fac}\, M$.

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] \arrow[l, dashed, bend right] & 3 \end{tikzcd}$$

$$\begin{tikzcd} & \frac{2}{3} \arrow[dr] & & \\ 3 \arrow[ur] \arrow[rr, dashed] & & 2 \arrow[dr] \arrow[rr, dashed] & & 1 \\ & & & \frac{1}{2} \arrow[ur] & \end{tikzcd}$$

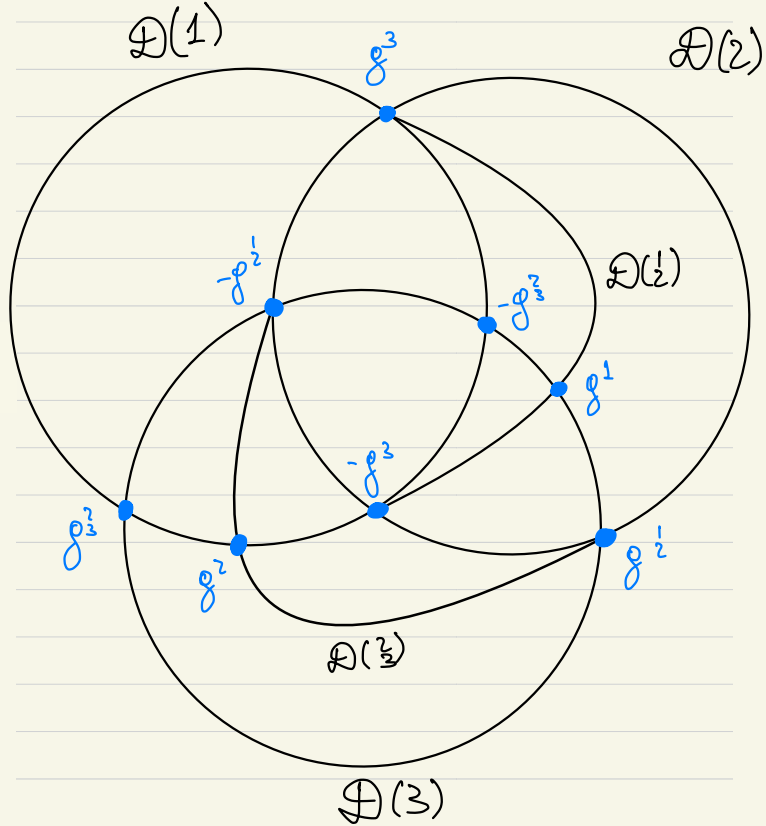
$$\mathcal{D}(1) = \{(0, y, z) : y, z \in \mathbb{R}\}$$

$$\mathcal{D}(2) = \{(x, 0, z) : x, z \in \mathbb{R}\}$$

$$\mathcal{D}(3) = \{(x, y, 0) : x, y \in \mathbb{R}\}$$

$$\mathcal{D}(\tfrac{1}{2}) = \{(x, -x, z) : x \geq 0, z \in \mathbb{R}\}$$

$$\mathcal{D}(\tfrac{2}{3}) = \{(x, y, -y) : x \in \mathbb{R}, y \geq 0\}$$



The first τ-Brauer-Thrall conj

Theorem [Demonet-Iyama-Jasso]

An algebra A is τ-tilting finite if and only if there are finitely many bricks in mod A.

Theorem [Schroll-T.]

An algebra A is τ-tilting finite if and only if

$$|\{ V \in \mathbb{Z}^n : V = [B] \quad B \text{ brick} \}| < \infty$$

Proof:

($\Rightarrow$) [DIJ]

($\Leftarrow$) $\mathcal{S} = \{ V \in \mathbb{Z}^n : V = [B] \quad B \text{ brick} \}$

$|\mathcal{S}| < \infty \Rightarrow |\{ \mathbb{H}_v \perp v : v \in \mathcal{S} \}| < \infty$

$\Rightarrow$ There are finitely many walls.

$\Rightarrow$ There are finitely many chambers

$\Rightarrow$ A is τ-tilting finite.

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Thank you very much!